Mathematics 244: Lab 1

Closed-form solutions

Fall 2005

Section 0: Introduction

In this lab we use Maple to find exact solutions of differential equations and initial value problems whether the solution is an explicit function or only defined implicitly.

Please turn in only the printout of your Maple worksheet. Include explicit answers to all questions asked, using the **text** feature of Maple to insert them in the worksheet. Remove from the worksheet any extraneous material and any errors you have made.

The first step is to obtain the **seed file** from the web page and arrange to save it in your directory on eden. You will be asked to execute the Maple commands that have been placed in this worksheet and to add some of your own. The ease of obtaining the seed file means that verbatim quotes of Maple instructions in the descriptions can be phased out, so not all prepared instructions will be described here.

In order to refer to results computed earlier in a worksheet, it is common to **assign** those results to a name. An alternative, introduced in Maple 10, is an **equation label** used to identify the last result in an **execution group**. When a result is needed, its equation number may be inserted through the menu bar of your Maple window, a context menu, or a keyboard shortcut (typically CTRL-L). The effect is to bring up a dialog box in which you enter the current label. These labels are automatically updated if commands are inserted or deleted.

The file begins with the line `with(plots):  with(DEtools):` in a preliminary section as in Lab 0.

Equation 1

We consider the equation

$$\frac{dy}{dx} + 5y = \frac{1}{1+e^x}, \tag{1}$$

The following commands (already entered into the seed files) show how Maple can find a general solution of the first order equation and to determine the value of the parameter appearing in that expression to get a solution with $y(0) = c$. Here, the solution y is found explicitly as a function of x (it is assumed that the solution will be given — as it was when this was tested — in the form of an **equation** involving $y(x)$ containing one arbitrary constant called `_C1`)

```
de1:= diff(y(x),x) + 5*y(x) = 1/(1 + exp(x));
s1:=dsolve(de1);
c1:=solve(eval(s1,{x=0,y(x)=c}),_C1);
s1a:=eval(s1,_C1=c1);
```

(a) Construction. This solution above is the same procedure you would use to solve the initial value problem if you were not using Maple. When you **are** using Maple, you can simply execute the following command to solve the initial value problem more simply and directly (note the use of braces { }).

```
s1b:=dsolve({de1,y(0)=c},y(x));
```

Execute this instruction and compare to the previous result. There should be no difference.

In either case, the solution produced by Maple is really a family of solutions, i.e., there is a different solution corresponding to each particular choice of the constant c. Solutions of first order equations contain one parameter. The quantity `_C1` playing that role in our first solution was essentially a **constant of integration** in the **method** used to solve the equation. In the second solution, we forced the parameter to be the **initial value** $y(0)$.

(b) Plots. The plots in this project will use instructions described directly in Maple that we combine into a single **display**. In this case, there is no difficulty giving a **complete description** of each component of the plot. If you want to test the individual `plot` instructions, or create them using **interactive tools**, use the supplementary worksheet. Your submitted worksheet should contain only the `plot` and `DEplot` instructions for constructing the individual plots and the `display` instruction for combining them.

The seed file contains statements to find the solutions `t1, t2, t3, t4, t5` corresponding to the choices $c = -2, -1, 0, 1, 2$, and construct a plot of those solutions on the same set of axes. The plot is not shown immediately, but saved to be shown as part of a composite display. Note that when naming the output of a plot, changing the ending semicolon to a colon will suppress output that shows details of the plot structure.

The second component of the display is a **direction field** constructed by the `DEplot` instruction (the complete instruction for this example appears in the seed file). The qualitative behavior of the differential equation can often be seen in such direction fields, although some practice is needed to interpret them. In this lab, direction fields will be used to check results found by other means.

Again, the instruction producing the named plot of the direction field ends with a colon to suppress output. The two plots are combined, and **displayed**, by the `display` command (part of the `plots` library). The instruction in the seed file includes a title. Every plot that appears in your report should have a title.

(c) Discussion. The plot suggests some features of the equation and its solutions that should be discussed. In particular, the picture suggests that, no matter what initial condition we start with, all solutions tend to the same value as $x \to \infty$. This can be verified using the formula `s1` for the general solution using `limit(s1,x=infinity);` (Maple considers `infinity` to be $+\infty$). **Answer in text**: What is the limit?

Now, try `limit(s1,x=-infinity);` and **interpret the answer in text** (you may need to use **Maple help** to identify terms used in the answer).

Some of this could be found **without solving the equation**. In particular, **what does the differential equation say** about where the solution $y(x)$ is **increasing**? **Answer in text**: Does this agree with your plot of the **direction field**?

You should also compare the results obtained by Maple with the solution that the textbook encourages you to use. The Maple instruction `odeadvisor(de1);` tells you how maple classifies the equation in order to select a solution method. Does the answer that Maple gives to this instruction agree with the description of that type of equation used in the textbook? Does the solution appear in the form expected from the method of solution proposed by the textbook for this type of equation?

In the remaining problems, an equation and some initial conditions will be given. Following the pattern of problem 1, **you should divide each problem into three parts**: (a) **Construction** of the solutions; (b) **Plots** of the solutions and a direction field; and (c) **Discussion** of aspects of the solution — especially comparing the Maple solution (as revealed by `odeadvisor`) with methods presented in the textbook. The step-by-step method used initially should not be used for the other equations.

Warning: In this lab description, the equations are written in the notation used in the textbook. This notation must be translated into the Maple language. In addition to the strict requirements on algebraic expressions in the **worksheet mode** used in these projects, Maple requires that a differential equation be described in terms of $y(x)$ consistently, you cannot abbreviate it to y. Use the solution of Problem 1 as a guide.

Equation 2

Introduce the name `de2` for the **equation**

$$\frac{dy}{dx} = \frac{x - e^{-x}}{5y + e^{y}}. \tag{2}$$

and use `dsolve` to solve the equation. You should name the result `s2`. In this case, you should find that the solution $y(x)$ is only defined **implicitly** as a function of x, i.e., the answer is an equation involving both $y(x)$ and x in which $y(x)$ is not solved explicitly as a function of x.

If the solution were given in the special form $f(x, y) = constant$, the `lhs` function would extract $f(x, y)$ to use as input to the `contourplot` command. Currently, the equation produced by `dsolve` is not given in this special form, so you should examine it to find the constant introduced into the solution — it is likely to be called `_C1` — and solve for it. An expression that accomplishes this, and gives a reasonable name to the result, is `V2:=solve(s2,_C1);` . This identifies an expression that is constant on solutions of the equation.

The second part of the problem is to construct a graph of solutions of the equation and combine it with the direction field produced by `DEplot`. Since we have a function that is constant on solutions of the equation, the graph of solutions may be constructed using `g2c:=contourplot(V2,x=-3..5,y=-2..3,color=black):` . The `display` command should be used to combine `g2` with the slope field produced by the `DEplot` command, and to **provide a title** for this plot.

For the third part, as with the previous equation, compare the result of `odeadvisor` with the classification in the textbook. Also, **answer in text**: many of the solutions of this equation appear to have a minimum or maximum at the same value of x; are these values really the same? Your answer should indicate a **feature of the equation** that supports your conclusion.

Equation 3

Consider the **logistic equation**

$$\frac{dy}{dt} = \frac{6}{5}y\left(1 - \frac{y}{3}\right). \tag{3}$$

As in previous exercises, use the instructions `dsolve`, `DEplot`, and `odeadvisor` to: **(a)** find the solution; **(b)** plot a slope field for $-3 \le t \le 3$ and $-1 \le y \le 6$ together with some solutions; and **(c)** interpret Maple's classification of the equation.

As part of your plot in part **(b)**, use appropriate instructions to obtain solutions with initial conditions $y(0) = 1$, 2, and 4. Plot graphs of these on the interval $-3 \le t \le 3$ and use the `display` command to combine this graph with the slope field obtained from the `DEplot` command. Be sure to **provide a title** for the combined plot.

In part **(c)**, describe connections between the report of `odeadvisor`, the form of the solution given by `dsolve`, and the method used in the text for solving this equation. In addition, **find the solution** of the equation with $y(0) = 6$ and evaluate that solution at $t = -2$. You should get $y(-2) = -0.6649543074$. Reconcile this negative value with the fact that the differential equation predicts that a solution through a point with $y > 5$ is a decreasing function with $y(t)$ **always** greater than 5.

Equation 4

Consider the equation

$$y + \left(2x - ye^{y}\right)\frac{dy}{dx} = 0, \tag{4}$$

which is essentially the equation of Exercise 21 of Section 2.6. As in previous problems, introduce `de4` as a name for the equation. Then, use the instructions `dsolve`, `DEplot`, and `odeadvisor` to **(a)** find the solution, **(b)** plot a slope field for $-2 \leq x \leq 2$ and $-1 \leq y \leq 3$ (just the slope field this time — no solutions), and **(c)** interpret Maple's classification of the equation.

The textbook informs you that this equation has an **integrating factor** $\mu(x, y) = y$. Maple knows this, too! The instruction `iF:=intfactor(de4);` will assign this integrating factor to the name `iF`. Then, use the instruction `de4a:=iF*de4;` to create a new equation that has been multiplied by this integrating factor. In your discussion in part **(c)**, describe any changes in the result of applying `dsolve` and `odeadvisor` to this equation instead of the original `de4`.