Mathematics 244: Lab 3

The Pendulum

Fall 2005

Section 0: Introduction

In this lab, we use Maple to examine differential equations modeling free oscillations. In particular, we shall consider linear and nonlinear models of a pendulum and compare the effect of damping in these models. Similar methods apply to **forced** oscillations, allowing the study of **resonance** and **beats**. For the linear equation, there are exact solutions that allow these properties to be studied via formulas. For the nonlinear equation, numerical solutions will be used. The supplementary worksheet contains some examples that can be used as a basis of other examples.

The first step is to obtain the **seed file** from the web page and save it in your directory on eden. You will be asked to execute the Maple commands that have been placed in this worksheet and to add some of your own. Please turn in only the printout of your main Maple worksheet. Include explicit answers to all questions asked, using the **text** feature of Maple to insert them in the worksheet. Remove any extraneous material from the worksheet. There is also a **supplementary worksheet** that suggests additional experiments that may lead to better answers to the questions considered in this project.

1. The linear model

Consider the motion of a pendulum, which consists of a mass m attached to one end of a rigid rod of length L. The other end of the rod is fixed at a point O and the rod is free to rotate about O. The position of the pendulum at time t is described by the angle $\theta(t)$ between the rod and the downward vertical direction, with the counterclockwise direction taken as positive (see Figure 9.2.2 on page 498 of the text). The differential equation governing the motion of the pendulum is

$$\frac{d^2\theta}{dt^2} + \frac{c}{mL}\frac{d\theta}{dt} + \frac{g}{L}\sin\theta = 0$$

It is expected that small changes in the equation will lead to small changes in the solution. Two modifications designed to approximate the equation by one that is more easily solved are to ignore damping by setting $c = 0$ or to replace $\sin\theta$ by θ to get a linear equation (or both). Note that θ is just the first term of the Taylor series for $\sin\theta$ about $\theta = 0$. As long as θ is **small**, so that the difference between $\sin\theta$ and θ is **very small**, the solutions to this linear equation should give a good approximation to those of the general equation (G). This project will show that, even when θ is not small, the qualitative aspects of the solution in the nonlinear case will be similar to the more easily determined properties of the linear case. In a mathematical treatment, only the values of the coefficients and not their expression in terms of physical measurements will enter, so we write the general equation as

$$\frac{d^2\theta}{dt^2} + \mu\frac{d\theta}{dt} + K\sin\theta = 0 \qquad (G)$$

and its linear approximation as

$$\frac{d^2\theta}{dt^2} + \mu\frac{d\theta}{dt} + K\theta = 0 \qquad (L)$$

This study is expected to take place on earth, where the constant g, the acceleration due to gravity, is 9.81m/sec^2. If we fix the length L at 1.09 meters, then $K = g/L = 9$. The examples that we consider will have values of the measurements m and c so that $\mu = c/(mL)$ takes on the values 1.5, 6, and 10. We also fix the initial conditions $\theta(0) = 0.75$, $\theta'(0) = 3.5$ corresponding to the pendulum being pushed vigorously and released $^3\!/_4$ radian from its rest position. These general definitions are collected in the following statements that appear at the start of both the **seed file** and the **supplementary worksheet**.

```
K:=9;
deG:=diff(theta(t),t,t) + mu*diff(theta(t),t)+K*sin(theta(t))= 0;
deL:=diff(theta(t),t,t) + mu*diff(theta(t),t)+K*theta(t)= 0;
Iv:=theta(0)=0.75, D(theta)(0)=3.5;
```

1a. Exact solution

The `dsolve` command can be used to compute the exact solution of the linear models. Its use is not exactly the same as for first order equations, since we now have two initial conditions. To solve equation (L) without damping and plot the solution, execute the following commands, which appear in both the **seed file** and the **supplementary worksheet**.

```
dom1:=t=0..10;
soln1a:=dsolve({eval(deL,mu=0),Iv});
gr1a:=plot(rhs(soln1a),dom1,color=BLUE):
```

Note that there is a statement ending with a colon where the **plot structure** `gr1a` is constructed without being shown , and this includes a choice of color for the graph. If different colors are used for different graphs, it will be easy to distinguish them when they are plotted on the same set of axes. The supplementary worksheet also contains the instruction `gr1a;` to show the graph that you have constructed.

1b. Damped motion

In the **supplementary worksheet**, modify the `dsolve` instruction used in part (a) to obtain solutions to the linear damped equations for the three cases $\mu = 1.5, 6, 10$. Use different names for all these solutions. Also, modify the `plot` instruction to **build graphs** of these solutions over the interval `dom1` using **different colors**. Do not **show** the graphs yet in your main worksheet. They will be shown in a combined `display` later. If any difficulty is revealed at that time, use the supplementary worksheet to explore the individual plots. Your main worksheet should contain only plots that you believe to be correct.

1c. Underdamping

Select the graph built in (b) that illustrates an **underdamped** system and combine it with the **undamped** graph from (a) and combine them in a single `display`. Be sure to give this graph a `title`. Then **compare** these two graphs in a **text discussion**, addressing the difference between the times when $\theta = 0$ for the two functions and the properties of the **critical points** (i.e., points where $d\theta/dt = 0$) on the two graphs.

1d. Overdamping

Select the graphs in (b) that illustrate **critical damping**, and **overdamping** and combine them in a single `display`. Be sure to give this graph a `title`. Then **compare** these two graphs in a **text discussion**, addressing the properties of the **critical points** (where $d\theta/dt = 0$) on the two graphs and the rates at which the two functions approach zero. In particular, which decreases faster: the **critically damped** or the **overdamped** example?

2. The nonlinear model

For the nonlinear equations, an **exact solution in elementary functions** cannot always be found. In such cases, the standard use of the `dsolve` command will not give a useful result. For this equation, different releases of Maple have given different results. The seed file contains `odeadvisor` and `dsolve` instructions for the both (G) and the equation obtained from it by setting $\mu = 0$. Test these command and **describe in text** whether it looks like these solutions are useful, including what you are able to learn from **Maple help**.

2a. Graphical options

The `DEplot` command works for higher order equations, but it shows only a graph (there is no direction field since it is a **second derivative** that is determined by the equation; there will be unique solutions corresponding to initial values of both a **point** and a **direction**, which means that there are solutions of **all** directions through each point). This command uses the classical fourth order Runge-Kutta method, so it will often be necessary to set a step size to get an accurate graph. These experiments should be done in the **supplementary worksheet**.

The **supplementary worksheet** contains the following instructions to illustrate the limitations described above. The first graph will have **visible flaws**, but a simple modification will produce a usable plot. The plot will be given a name for later use, and then displayed by entering that name as a command.

```
DEplot(eval(deG,mu=0),theta(t),t=0..10,{[Iv]});
opt2:=linecolor=GREEN,stepsize=0.3,thickness=2;
gr2:=DEplot(eval(deG,mu=0),theta(t),t=0..10,{[Iv]},opt2):
gr2;
```

Most of the options in `opt2` only serve cosmetic purposes: choosing a color and thickness of the curve instead of accepting Maple's choices, but the choice of `stepsize` determines the accuracy of the computation. The value in `opt2` is good enough that you should **see the difference**. The supplementary worksheet also contains another option in which the `stepsize` has been cut by a factor of 3, and the cosmetic options changed to allow the graphs to be distinguished. A `display` command is used to graph the two solutions on the same axes. This change in the `stepsize` triples the time required to compute the graph, so it should only be done if the results are significantly different. Since a **fourth order** method is used, cutting the `stepsize` by another factor of 3 would decrease the error by a factor of 81, so this graph is also able to predict whether any further refinement of the `stepsize` would be useful (the use of a factor of 3 here is arbitrary). After comparing the two given options, and perhaps experimenting with other values, **choose a stepsize** that you feel will give the **best balance** between **computer effort** and **accuracy of the result**, add appropriate cosmetic options, and save this under the name `opt2` in your main worksheet. Follow this with a **discussion** explaining your choices. You should **refer** to the results of experiments in the supplementary worksheet, but you do not need to include the graphs constructed there.

In each of the remaining parts, you should use the options selected in this part to construct a graph of a numerical solution to the nonlinear equation (G) corresponding to a solution of (L) found in part 1. For each choice of μ, the two graphs should be combined in a single `display`. You should then **discuss** the differences between the graphs in **text**. Suggestions for that discussion are given in the individual sections below.

2b. Undamped motion

For $\mu = 0$, both equations should have **periodic** solutions. **Compare the period** of the solutions of the two equations. Also, **compare the maximum value** of the solutions of the two equations.

2c. Underdamped motion

These solutions repeatedly return to $\theta = 0$. In the linear case, the time between these returns is easily seen to be regular. Does this appear to also be true in the nonlinear case? Also, **compare** the time between returns for the two graphs.

2d. Critically damped motion

One important feature of this case are the **maximum value** of θ and the time at which it attained. **Compare** these between the solutions of the linear and nonlinear equations.

2e. Overdamped motion

Does the decay of the solution in the nonlinear case resemble the **exponential decay** of the linear case? **Compare** the rate of decay.

Note: The supplementary worksheet suggests some experiments for investigating **forced motion** in these models. For these experiments, the initial conditions will be $\theta'(0) = \theta(0) = 0$. Although this leads to the trivial solution $\theta(t) = 0$ in the homogeneous case, there is nothing special about these initial conditions in the inhomogeneous case, so they are usually used to provide consistent initial conditions when comparing different driving forces.

Since this was only announced after the project was assigned in some sections, there will be a **required** response to those experiments only if your instructor announces one. These experiments illustrate the phenomena of **resonance** and **beats**. In the linear case, there are **exact solutions** that allow this kind of information to be extracted from formulas. Numerical solutions allow solutions of the nonlinear equation to be examined for similar behavior. Additional questions that may be considered are:

(1) If a nonlinear system is driven at the **resonant frequency**, do the solutions exhibit increasing amplitude in the same way as solutions to the linear equation?

(2) How can the difference between the driving frequency and the resonant frequency be seen in the graph of the solution?

(3) Is there a **different** frequency that seems more resonant?

(4) For a linear equation, changing the amplitude of the driving force produces a corresponding change in the resulting motion. Is this also true for the nonlinear equation or are there **qualitative** changes in the solution as the amplitude changes?

If answers to these items are required, your discussion should be supplemented with a small number of graphs. These graphs should be included by copying instructions for generating them from the supplementary worksheet to the main worksheet.

End of Lab 3