Mathematics 244 Essay 3

Inhomogeneous Systems

Fall 2005

0. Introduction and Setup

An inhomogeneous linear system has the form

$$\frac{d\mathbf{y}}{dt} = M\mathbf{y} + \mathbf{b}, \tag{1}$$

where M is an n by n matrix of functions of t, although most of our discussion will be devoted to the case of a constant matrix, and $\mathbf{b}$ is a vector with n entries, each a function of t. This may be applied to a single higher order equation by converting the equation to a system. General methods, as well as all theoretical considerations, are better developed in the setting of systems, but there may be special methods that allow simpler computations when applied to single higher order equations.

The general method for linear systems is to find **one particular solution** of (1) and add to it a **general solution of the homogeneous equation** with the same M obtained by setting $\mathbf{b} = \mathbf{0}$ in (1).

We assume that the **independent variable** t has been translated so that we will be given **initial conditions** at $t = 0$, and that $t = 0$ is an **ordinary point** of the system, i.e., a point at which the hypotheses of the existence and uniqueness theorems hold. Then, a **fundamental matrix** $\Phi(t)$ is a matrix such that

$$\frac{d\Phi(t)}{dt} = M\Phi(t) \quad \text{and} \quad \det \Phi(0) \neq 0.$$

In the constant coefficient case, it is convenient to require that $\Phi(t) = e^{Mt}$. That is, to assume that $\Phi(0)$ is an identity matrix. This matrix has many useful properties and can be easily determined in many cases. If M is not constant, this additional normalization is less useful, so we prefer to find a fundamental matrix that has other simplifying features (an example will be given later).

1. Variation of parameters

The method for solving the general first order linear equation extends to systems, where it is called **Variation of Parameters**. Once one has a fundamental matrix Φ, the substitution $\mathbf{y} = \Phi\mathbf{v}$ introduces a new variable $\mathbf{v}$ that satisfies a **much simpler** equation. Indeed,

$$\Phi\frac{d\mathbf{v}}{dt} + \left(\frac{d}{dt}\Phi\right)\mathbf{v} = \frac{d\mathbf{y}}{dt} = M\mathbf{y} + \mathbf{b} = M\Phi\mathbf{v} + \mathbf{b}$$

Since

$$\left(\frac{d}{dt}\Phi\right) = M\Phi,$$

it follows that

$$\Phi\frac{d\mathbf{v}}{dt} = \mathbf{b}$$

or

$$\frac{d\mathbf{v}}{dt} = \Phi^{-1}\mathbf{b}. \tag{2}$$

Once Φ^{-1} is available, $\mathbf{v}$ is found by integration, and $\mathbf{y}$ is found by multiplying by Φ. The **simple computation** leading to (2), and from there to $\mathbf{v}$ and the solution $\mathbf{y}$ is so easy to reproduce that one should resist the temptation to summarize it by a **less natural** formula.

2. Constant coefficients

The matrix

$$M = \begin{bmatrix} 5 & -4 \\ 3 & -3 \end{bmatrix}$$

appeared as **Example 1** in the **Matrix Exponentials** essay. We found

$$\begin{aligned} e^{Mt} &= \frac{1}{4}\big((M+I)e^{3t} - (M-3I)e^{-t} \big) \\ &= \frac{1}{4}\begin{bmatrix} 6 & -4 \\ 3 & -2 \end{bmatrix} e^{3t} + \frac{1}{4}\begin{bmatrix} -2 & 4 \\ -3 & 6 \end{bmatrix} e^{-t} \end{aligned}$$

Although this matrix was originally written in terms of **hyperbolic functions**, it is the expression in terms of exponentials that will be useful here. We know from other considerations that the exponents are λt, where λ is an **eigenvalue** of M, and that the columns of the matrix multiplying this exponential are **eigenvectors** of M with eigenvalue λ. Much more is true: if $e^{Mt} = P_1 e^{\lambda_1 t} + P_2 e^{\lambda_2 t}$, then the P_i are **projection matrices**. It is immediate that $P_1 + P_2 = I$ (from the initial condition), and $\lambda_1 P_1 + \lambda_2 P_2 = M$ (from the differential equation), but we have also seen that each P_i is a **projection**, so that $P_i^2 = P_i$. This requires that $P_1 P_2 = \mathbf{0}$. Such P_1 and P_2 will be called **complementary projections**. Systems in higher dimensions also have this **splitting by projections**, but more effort is usually required to find it.

One consequence of the splitting by projections is that e^{-Mt} can be found **immediately** once e^{Mt} is available. The formula $e^{-Mt} = P_1 e^{-\lambda_1 t} + P_2 e^{-\lambda_2 t}$ is easily verified by multiplying this expression with the splitting of e^{Mt} by projections.

If $\mathbf{b} = \mathbf{c}e^{\alpha t}$ where $\mathbf{c}$ is a **constant vector**, then

$$e^{-Mt}\mathbf{b} = P_1 \mathbf{c} e^{(\alpha - \lambda_1)t} + P_2 \mathbf{c} e^{(\alpha-\lambda_2)t} \tag{3}$$

and each term is easily integrated.

3. An example

Consider

$$\frac{d\mathbf{y}}{dt} = \begin{bmatrix} 5 & -4 \\ 3 & -3 \end{bmatrix}\mathbf{y} + \begin{bmatrix} -2 \\ 5 \end{bmatrix} e^{2t}$$

Then, with $\mathbf{y} = e^{Mt}\mathbf{v}$, (3) yields

$$\frac{d\mathbf{v}}{dt} = \frac{1}{4}\begin{bmatrix} 6 & -4 \\ 3 & -2 \end{bmatrix} e^{-3t}\begin{bmatrix} -2 \\ 5 \end{bmatrix} e^{2t} + \frac{1}{4}\begin{bmatrix} -2 & 4 \\ -3 & 6 \end{bmatrix} e^{t}\begin{bmatrix} -2 \\ 5 \end{bmatrix} e^{2t}$$

Thus,

$$\mathbf{v} = \left(\frac{-1}{4}\begin{bmatrix} 6 & -4 \\ 3 & -2 \end{bmatrix} e^{-t} + \frac{1}{12}\begin{bmatrix} -2 & 4 \\ -3 & 6 \end{bmatrix} e^{3t} \right)\begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

The multiplication of $\mathbf{c}$ by the projection matrices is not performed at this point to emphasize the simple rule for building $\mathbf{y}$ from this expression for $\mathbf{v}$. Since the matrices appearing in this expression are **complementary projections**, multiplication by e^{Mt} only modifies the exponents in each term to get

$$\begin{aligned} \mathbf{y} &= \left(\frac{-1}{4}\begin{bmatrix} 6 & -4 \\ 3 & -2 \end{bmatrix} e^{2t} + \frac{1}{12}\begin{bmatrix} -2 & 4 \\ -3 & 6 \end{bmatrix} e^{2t} \right)\begin{bmatrix} -2 \\ 5 \end{bmatrix} \\ &= \left(\begin{bmatrix} 8 \\ 4 \end{bmatrix} e^{2t} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{2t} \right) \\ &= \begin{bmatrix} 10 \\ 7 \end{bmatrix} e^{2t} \end{aligned}$$

4. A more difficult example

If $\mathbf{b} = \mathbf{c}e^{\lambda t}$ where λ is an eigenvalue of M, something different happens — one of the terms in $\mathbf{v}$ is the integral of a **constant vector**, so a factor of t is introduced. When $\mathbf{y}$ is found by multiplying by e^{Mt}, this term becomes **an eigenvector** for λ times $t^{\lambda t}$ while the other term is **an eigenvector** for **the other eigenvalue** times $e^{\lambda t}$. The scaling of these eigenvectors is determined by $\mathbf{c}$.

Using the same M as before, but with

$$\mathbf{b} = \begin{bmatrix} 2 \\ -7 \end{bmatrix} e^{3t}$$

and following the same steps as before,

$$\begin{aligned}
\frac{d\mathbf{v}}{dt} &= \frac{1}{4}\begin{bmatrix} 6 & -4 \\ 3 & -2 \end{bmatrix} e^{-3t}\begin{bmatrix} 2 \\ -7 \end{bmatrix} e^{3t} + \frac{1}{4}\begin{bmatrix} -2 & 4 \\ -3 & 6 \end{bmatrix} e^{t}\begin{bmatrix} 2 \\ -7 \end{bmatrix} e^{3t} \\
\mathbf{v} &= \left(\frac{1}{4}\begin{bmatrix} 6 & -4 \\ 3 & -2 \end{bmatrix} t + \frac{1}{16}\begin{bmatrix} -2 & 4 \\ -3 & 6 \end{bmatrix} e^{4t}\right)\begin{bmatrix} 2 \\ -7 \end{bmatrix} \\
\mathbf{y} &= \left(\frac{1}{4}\begin{bmatrix} 6 & -4 \\ 3 & -2 \end{bmatrix} te^{3t} + \frac{1}{16}\begin{bmatrix} -2 & 4 \\ -3 & 6 \end{bmatrix} e^{3t}\right)\begin{bmatrix} 2 \\ -7 \end{bmatrix} \\
&= \left(\begin{bmatrix} 10 \\ 5 \end{bmatrix} te^{3t} + \begin{bmatrix} -2 \\ -3 \end{bmatrix} e^{3t}\right)
\end{aligned}$$

5. Undetermined coefficients

If α is not an eigenvalue of M, this method yields $\mathbf{y} = \mathbf{k}e^{\alpha t}$ for some **constant vector** $\mathbf{k}$. This **fact** can be used to solve the equation **without finding the projections** P_1 and P_2. It is only necessary to substitute the known form of $\mathbf{y}$ into the differential equation to get an equation in $\mathbf{k}$. For the original example, this is $(2I - M)\mathbf{k}e^{2t} = \mathbf{b}$, or

$$\begin{bmatrix} -3 & 4 \\ -3 & 5 \end{bmatrix}\mathbf{k} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

This can be solved by left multiplication by

$$\begin{bmatrix} -3 & 4 \\ -3 & 5 \end{bmatrix}^{-1} = \frac{-1}{3}\begin{bmatrix} 5 & -4 \\ 3 & -3 \end{bmatrix}.$$

6. Inverses of two-by-two matrices

Although you are told in a **Linear Algebra** course that systems of linear equations should **always** be solved by **Gaussian elimination**, you are **tricked** into believing this by being given exercises that are large systems in which the steps always involve matrices with integer entries (except possible some **tame fractions** in the last step). Without careful selection of the exercises, the proposed method of solution may fail to be suitable for hand computation. For 2-by-2 (and perhaps also 3-by-3) systems, a direct computation of the inverse — equivalent to Cramer's rule — may be the best method to solve the equations. The general formula, which is easily checked, is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

7. Higher order equations

To solve a linear differential equation with constant coefficients, written in the form $L[y] = ce^{\alpha t}$, using a **linear differential operator** L to represent operations performed on y on the left side of the equation, **undetermined coefficients** should be used. Indeed, since L is a **linear operator**, the undetermined coefficients themselves need not be written. It is only necessary to find $L[e^{\alpha t}]$. Since L has constant coefficients, it is always the case that $L[e^{\alpha t}] = pe^{\alpha t}$ for some constant p. If $p \neq 0$, linearity guarantees that $L[(c/p)e^{\alpha t}] = ce^{\alpha t}$, so we have a particular solution to the equation. The general solution is found by adding the general solution of the corresponding homogeneous equation to this particular solution.

If $L[e^{\alpha t}] = 0$, one should find $L[te^{\alpha t}]$ to continue the search for a solution of the equation. If also $L[te^{\alpha t}] = 0$, one should find $L[t^2 e^{\alpha t}]$, etc. The first nonzero result will be of the form $pe^{\alpha t}$.

For such equations, the undetermined coefficient method leads directly to a solution, so no other method should be considered.

8. More complicated equations and systems

If the right side of an equation involves $\sin \beta x$ and/or $\cos \beta x$, one should start by computing **both** $L[\sin \beta x]$ and $L[\cos \beta x]$. The combination yielding the given right side can be found by solving a system of two equations.

If the right side contains several unrelated terms, solutions for each term should be found separately, and the complete solution will be the sum of these pieces and the general solution to the homogeneous equation. It is not necessary to determine the form of the solution in advance, every time you find a piece of the solution, the quantity obtained by subtracting this piece from y satisfies a **simpler equation**.

An important special case has a polynomial of degree n on the right side. Normally, $L[t^n]$ will be a polynomial of degree n, so there will be a constant a_n so that $L[y - a_n t^n]$ will be a polynomial of degree **strictly less than** n. This is simpler, and repeating this step will find all terms of a polynomial solution of the equation.

9. Systems with variable coefficients

The equation

$$(t-1)\frac{d^2 y}{dt^2} - t\frac{dy}{dt} + y = 0 \tag{4}$$

is easily seen to have $y = t$ and $y = e^t$ as solutions, although there may be no reasonable way to **discover** these solutions from the equation alone. This means that

$$\Phi = \begin{bmatrix} t & e^t \\ 1 & e^t \end{bmatrix}$$

is a **fundamental matrix** of the system

$$\frac{d}{dt}\mathbf{y} = \frac{1}{t-1}\begin{bmatrix} 0 & t-1 \\ -1 & t \end{bmatrix}\mathbf{y}$$

obtained from (4) by the usual procedure of letting $\mathbf{y}$ be the vector with entries y and y' and using (4) to express the derivatives of these quantities as linear combinations of y and y'.

Using the formula for the inverse,

$$\Phi^{-1} = \frac{1}{t-1}\begin{bmatrix} 1 & -1 \\ -e^{-t} & te^{-t} \end{bmatrix}$$

To solve

$$\frac{d}{dt}\mathbf{y} = \frac{1}{t-1}\begin{bmatrix} 0 & t-1 \\ -1 & t \end{bmatrix}\mathbf{y} + \begin{bmatrix} 0 \\ t-1 \end{bmatrix}$$

we find

$$\mathbf{v}' = \Phi^{-1}\begin{bmatrix} 0 \\ t-1 \end{bmatrix} = \begin{bmatrix} -1 \\ te^{-t} \end{bmatrix}$$

$$\mathbf{v} = \begin{bmatrix} -t \\ -(t+1)e^{-t} \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} -t^2 - t - 1 \\ -2t - 1 \end{bmatrix}$$

Note that **b** betrays its origin in a second order equation by having zero in its first entry, since the first equation of the system must continue to assert that $dy/dt = y'$. Although **v** and its derivative, arising in the work, have no special properties, the vector **y** has its second entry equal to the derivative of its first entry as required by the first equation of the system. It is this property of the solution that makes it preferable to work with single higher order equations rather than systems in some cases. Even this example would admit an easy solution by undetermined coefficients **if we knew** that there was a polynomial solution. (A search for a **series solution** would give some kind of solution in all cases, and would discover polynomial solutions when they were present.)

10. Exercises

Find **one particular solution** to each of the following using both **variation of parameters** and **undetermined coefficients**. You should get the same answer from both methods. The variation of parameters method requires finding a complete solution of the corresponding homogeneous equation, so you can combine this with your particular solution to get a **general** solution of each equation.

1. $$\frac{d\mathbf{y}}{dt} = \begin{bmatrix} 11 & 8 \\ -4 & -1 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 1 \\ 5 \end{bmatrix}e^{4t}$$

2. $$\frac{d\mathbf{y}}{dt} = \begin{bmatrix} 11 & -3 \\ 18 & -4 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 7 \\ -3 \end{bmatrix}e^{3t}$$

3. $$\frac{d\mathbf{y}}{dt} = \begin{bmatrix} 14 & 6 \\ -30 & -13 \end{bmatrix}\mathbf{y} + \begin{bmatrix} -5 \\ 1 \end{bmatrix}$$

4. $$\frac{d\mathbf{y}}{dt} = \begin{bmatrix} 28 & 12 \\ -56 & -24 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 2 \\ 5 \end{bmatrix}e^{3t}$$

5. $$\frac{d\mathbf{y}}{dt} = \begin{bmatrix} 5 & -9 \\ 8 & -7 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 8 \\ -1 \end{bmatrix}e^{-2t}$$