Remember that no books, notes, or calculators may used during an actual exam.

On the exam you will be given these formulas:

Integrating factor

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu$$

Improved Euler

$$k_1 = f(t_n, y_n)$$
$$k_2 = f(t_{n+1}, y_n + h\,k_1)$$
$$y_{n+1} = y_n + (h/2)(k_1 + k_2)$$

Runge-Kutta

$$k_1 = f(t_n, y_n)$$
$$k_2 = f(t_n + h/2, y_n + (h/2)k_1)$$
$$k_3 = f(t_n + h/2, y_n + (h/2)k_2)$$
$$k_4 = f(t_{n+1}, y_n + h\,k_3)$$
$$y_{n+1} = y_n + (h/6)(k_1 + 2k_2 + 2k_3 + k_4)$$

1. Solve the following initial value problems; if possible, give an explicit solution and determine the maximum interval on which the solution is defined.

(a) $\cos x + 2xe^y + (x^2+2)e^y y' = 0$; $y(0) = 1$. (b) $3\dfrac{y}{x} - x + y' = 0$; $y(1) = -1$.

(c) $y'' + 5y' + 6y = 0$; $y(1) = 2$, $y'(1) = 0$. (d) $y' = 2x(1+y^2)$; $y(1) = 0$.

2. Find the general solution to these differential equations; give an explicit solution if possible.

(a) $3x^2y + 2xy + y^3 + (x^2+y^2)y' = 0$. (b) $y'' - 6y' + 25y = 0$.

3. Find a fundamental set of solutions for the equation in problem 2(c). Use the Wronskian to verify that they are indeed a fundamental set.

4. (a) Define what it means for two functions $y_1(x)$ and $y_2(x)$, defined on an interval $a < x < b$, to be *linearly independent* there.

(b) Suppose that L is an operator acting on functions defined on this interval (that is, if $y(x)$ is a function defined for $a < x < b$, so is $L[y]$). What does it mean for L to be *linear*?

(c) Suppose that L is linear and that $y_1(x)$, $y_2(x)$ satisfy $L[y_1] = L[y_2] = 0$. Show that also $L[c_1y_1 + c_2y_2] = 0$ for any constants c_1, c_2.

(d) We define L (on any interval) by $L[y] = y' - y + 5$. Is L linear? Justify your answer.

5. A certain quantity $y(t)$ satisfies the differential equation $y' = (y-2)(e^y - 1)$.

(a) Find all equilibrium solutions and classify each as stable, semi-stable, or unstable.

(b) Find $\lim_{t\to\infty} y(t)$ if (i) $y(0) = 1$; (ii) $y(0) = 0$; (ii) $y(0) = 4$.

6. For the initial value problem $y' = -y$, $y(0) = 2$, estimate $y(1)$ using the Euler, improved Euler, and Runge-Kutta methods, each with $h = 1$. (That is, do one step of each. This is *not* a realistic problem!)

7. In the Runge-Kutta method the local truncation error is known to be of order h^5, where h is the step size. (a) Explain why this gives a global error of order h^4. (b) If a Runge-Kutta computation, with step size 0.1, of the solution of some specific problem $y' = f(t,y)$, $y(0) = y_0$, gives an error of 0.0100 in the approximate value of $y(2)$ found, approximately what will be the error if a step size of 0.02 is used?

8. A certain water tank has the shape of a cube of side 10 ft. The tank is full at noon on February 16, 1996; after that, water is added at the rate of 1 cubic foot per minute and leaks out at a rate proportional to the depth of water in the tank. (The initial leakage rate is more than 1 cubic foot per minute so that no water overflows.)
(a) Give a differential equation for the rate of change of the depth of water. Justify briefly.
(b) It is observed that after a *very* long time (days and days and days) the depth of water in the tank is 2 ft and can no longer be seen to be changing. Determine the amount of water in the tank at 1:00 P.M. on February 16, 1996.

9. For each of the initial value problems (a), (b), and (c) below, answer these questions:
(i) Do the existence and uniqueness theorems we discussed imply that a unique solution exists on some interval $a < x < b$ (with $a < 0$ and $b > 0$)?
(ii) If so, do the theorems allow you to determine specific values for a and b, and if so, what are these values?
Give a *brief* explanation of your reasoning in each case. Do *not* solve the equations.
(a) $y' = x^2 + y^2 \cos x + 1, \quad y(0) = 0.$ (b) $y' = x + y \cos x + 1, \quad y(0) = 0.$
(c) $y' = \sqrt{x} + \sqrt{y} \cos x + 1, \quad y(0) = 0.$

10. In (a)–(e) we give four descriptions of direction fields in the xy plane. For each of these, choose the differential equation from the list (i)–(ix) whose direction field would satisfy the description.
(a) The direction field is unchanged if shifted up or down.
(b) The direction field is unchanged if shifted left or right.
(c) The direction field is horizontal on the line of slope 1 through the origin.
(d) The direction field on the line of slope 1 through the origin is parallel to this line.
(e) The direction field slopes up in the first and third quadrants, down in the second and fourth.
(i) $y' = xy - 1$; (ii) $y' = x^3/y$; (iii) $y' = x \cos x$; (iv) $y' = ye^x$; (v) $y' = ye^y$;
(vi) $y' = (x + 2)/(y + 1)$; (vii) $y' = \cos(y - x)$; (viii) $y' = xy \sin(y - x)$; (ix) $y' = x + y$.

Brief answers (not checked too carefully—be skeptical). For maximum benefit from the review problems you should work the problem until you get the right answer, not work backward from the answer or just check that the answer looks correct.

1(a) $y = \ln\left(\dfrac{2e - \sin x}{x^2 + 2}\right)$, $(-\infty, \infty)$; (b) $y = \dfrac{x^2}{5} - \dfrac{6}{5x^3}$, $(0, \infty)$;
(c) $y = 6e^{-2x+2} - 4e^{-3x+3}$, $(-\infty, \infty)$; (d) $y = \tan(x^2 - 1)$, $(-1.60337, 1.60337)$.
2(a) $(3x^2y + y^3)e^{3x} = c$; (b) $e^{3t}(A \cos 4t + B \sin 4t)$.
3 $y_1 = e^{3t} \cos 4t$, $y_2 = e^{3t} \sin 4t)$, $W(t) = 4e^{6t} \neq 0$..
4(a)–(c) See book or lecture notes; (d) No.
5(a) $y = 0$ is stable; $y = 2$ is unstable; (b) 0, 0, ∞.
6 Euler: 0; Improved Euler: 1; Runge-Kutta: 0.75; Exact: 0.736.
7(a) See book or lecture notes; (b) 0.000016.
8(a) $dD/dt = -kD + 1/100$; (b) $D(60) = 8e^{-3/10} + 2$.
9(a) Yes, no; (b) Yes, yes: $-\infty < x < \infty$; (c) No.
10 (a) (iii); (b) (v); (c) (viii); (d) (vii); (e) (ii).