

Remember that no books, notes, or calculators may be used during an actual exam. You will be given these formulas:

$$\begin{aligned} \text{Weight (pounds)} &= \text{mass (slugs or lb-sec}^2/\text{ft)} \times g \text{ (32 ft/sec}^2\text{)}; \\ u'_1 &= \frac{-y_2 g}{W[y_1, y_2]} & u'_2 &= \frac{y_1 g}{W[y_1, y_2]} & v'' + (p + 2\frac{y'_1}{y_1})v' &= 0. \end{aligned}$$

1. Find the general solution of $y'' - 2y' + 2y = x^2 + 2 + 2\cos x$ by the method of undetermined coefficients. Express your answer in terms of real quantities.

2. Find the general solution of the equation $xy'' - (x+1)y' + y = 0$. Hint: $y = e^x$ is one solution.

3. Find the general solution of the equation $x^2y'' + 2xy' - 6y = x^2$.

4. Solve the initial value problem for $\mathbf{x}(t)$: $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$, where $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 5 & 6 \\ 0 & -3 & -4 \end{pmatrix}$.

5. Let $A = \begin{pmatrix} 1 & -3 \\ 2 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -1 & 0 \\ 1 & -3 & 4 \end{pmatrix}$, and $\mathbf{b} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Compute AB and solve $A\mathbf{x} = \mathbf{b}$.

6. (a) A mass of 1 slug (weight 32 lb) hangs from a spring of constant 8 lb/ft; we ignore friction. The mass is released from 1 ft above its equilibrium position with a downward speed of 6 ft/sec. Find the subsequent motion and determine (i) the first time that mass reaches its equilibrium position, and (ii) how far down the mass goes.

(b) The system is now placed in heavy oil, introducing a damping constant of 6 lb-sec/ft, and the mass is released in the same way. Determine whether or not the equilibrium position is reached in finite time.

7. Solve $y'' - 6y' + 9y = 0$; $y(0) = 1$, $y'(0) = -2$

8. Determine whether or not the three vectors $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} -1 \\ 6 \\ 1 \end{pmatrix}$, $\mathbf{x}_3 = \begin{pmatrix} -1 \\ 10 \\ 5 \end{pmatrix}$ are linearly independent. Justify your answer.

9. Find the general solution of the equations $x' = x + y$; $y' = 4x - 2y$, and give a careful drawing of the phase plane (xy -plane) for this system, showing various typical and special trajectories.

10. Redo exercises 2, 3, and 4 from the *Notes on Linear Algebra* on the web page.

11. Let $\mathbf{x}' = P\mathbf{x}$ be a system of equations for the unknown function $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, where $P = P(t)$ is a 2×2 matrix whose coefficients are functions of t which are defined and continuous for all t . Let $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ be two solutions of this system. (a) Show that $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$ is also a solution, for c_1 and c_2 constants. (b) Define the Wronskian $W(t)$ of $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$. (c) Show that if $W(0) \neq 0$ then a solution of the form given in (a) can be used to solve any initial value problem $\mathbf{x}' = P\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}^0$.

ANSWERS (not checked too carefully—be wary):

1. $y(x) = Ce^x \cos x + De^x \sin x + x^2/2 + x + 3/2 + (2/5)\cos x - (4/5)\sin x$.

2. $y(x) = c_1e^x + c_2(x+1)$. 3. $y(x) = c_1x^{-3} + c_2x^2 + (1/5)x^2 \ln x$.

4. $\mathbf{x}(t) = -11e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 4e^{-t} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - 3e^{2t} \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$. 5. $\begin{pmatrix} -1 & 8 & -12 \\ 8 & -14 & 16 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} -1/5 \\ -2/5 \end{pmatrix}$.

6. (a) $u(t) = -\cos 2\sqrt{2}t + (6/2\sqrt{2})\sin 2\sqrt{2}t$; $t_0 = (1/2\sqrt{2})\arctan(\sqrt{2}/3)$. (b) Yes, at time $t = \ln(\sqrt{2})$.

7. $e^{3t} - 5te^{3t}$. 8. No. 9. $c_1e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2e^{-3t} \begin{pmatrix} 1 \\ -4 \end{pmatrix}$; saddle point.

10. See notes for some answers. 11. See book or class notes.