The exam will be held **Wednesday, December 20, 12:00-3:00 PM, in Hill 116.** No books, notes, or calculators may used during the exam.

The exam will not include the material in Section 2.6 on integrating factors; the material in Section 2.6 on exact equations will be included. The exam will not include Section 7.7 on fundamental matrices and the matrix exponential.

You will be given the following formula sheet.

Formulas

Integrating factor: $\mu = e^{\int p(x)\,dx}$.

Reduction of order: $v'' + \left(p + 2\frac{y_1'}{y_1}\right)v' = 0$.

Variation of parameters: $u_1' = \frac{-y_2 g}{W(y_1, y_2)}$, $u_1' = \frac{y_1 g}{W(y_1, y_2)}$.

Complex numbers: $e^{i\theta} = \cos\theta + i\sin\theta$.

Infinite series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad \frac{1}{(1-x)} = 1 + x + x^2 + x^3 + \cdots$$

English units: Weight (in pounds) = mass (in slugs) $\times$ g (32ft/sec^2).

This review sheet contains primarily problems on material covered since the second exam, although there are a few problems on earlier material. However, the exam will be cumulative. **For a complete review you should use also the problem sheets for exams 1 and 2**, still available on the web page; reviewing exams 1 and 2 themselves may also be useful.

1. Consider the equations $x' = x(x + y - 4)$, $y' = y(1 + 2x - y)$.

(a) Show that $x = 1$, $y = 3$ is a critical point, find the linearized equations near this point, and thus determine whether or not this critical point is stable, asymptotically stable, or unstable. What is the *type* of this critical point?

(b) Find all other critical points.

(c) Sketch the first quadrant $x \geq 0$, $y \geq 0$ of the phase plane, indicating, by arrows or otherwise, regions where x and y are increasing, x is increasing and y decreasing, etc., and where the trajectories are horizontal and vertical.

(d) For each initial condition below, determine (from your sketch) $\lim_{t\to\infty}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$:

(i) $x(0) = 0$, $y(0) = 3$; (ii) $x(0) = .5$, $y(0) = 2$.

2. Consider the system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{pmatrix} 0 & -1 \\ 5 & 2 \end{pmatrix}$. (a) What is the type and stability of the critical point at the origin? (b) Find a solution (in terms of real functions) which satisfies $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

3. (a) Find the general solution of the equations $x' = 3x - y$; $y' = 4x - 2y$. (b) Give a careful drawing of the phase plane (xy-plane) for this system, showing various typical and special trajectories.

(c) Find initial conditions (x_0, y_0) such that if a solution satisfies $x(0) = x_0$, $y(0) = y_0$ then

(i) $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = 0$; (ii) $\lim_{t\to-\infty} x(t) = \lim_{t\to-\infty} y(t) = 0$;

(iii) $\lim_{t\to\infty} x(t) = \lim_{t\to-\infty} x(t) = -\infty$.

4. (a) Solve $y'' + 2xy' + 4y = 0$ as a series $y = \sum_{n=0}^{\infty} a_n x^n$, finding all coefficients up to order x^7 and displaying your final answer in the form $y(x) = a_0 y_0(x) + a_1 y_1(x)$.

(b) Find the solution of the equation which satisfies $y(0) = 3$, $y'(0) = 1$.

(c) Find the general formula for a_n and then find either y_0 or y_1 in closed form—i.e., sum the corresponding series.

5. The equation $P(x)y'' + x^2y' + (x^5 - 1)y = 0$ is to be solved in a power series of the form $y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$. For each case below, determine whether x_0 is an ordinary point (so that such a solution is possible) and, if it is, find the minimum possible radius of convergence of the resulting series.

(i) $P(x) = \cos x$, $x_0 = 0$;
(ii) $P(x) = x^2 - 2x + 5$, $x_0 = 3$;
(iii) $P(x) = e^x$, $x_0 = 10$;
(iv) $P(x) = \sin x$, $x_0 = 0$.

6. (From Exam 2): In (a) and (b) below give a matrix R in row-echelon form satisfying the given condition, or explain why this is impossible.

(a) R is a 4×3 matrix. The equations $R\mathbf{x} = 0$ have a unique solution.

(b) R is a 4×4 matrix. For every $\mathbf{b}$ the equations $R\mathbf{x} = \mathbf{b}$ have a solution containing a free parameter.

7. A certain time-dependent physical variable y is believed to satisfy the differential equation $dy/dt = \sin y$.

(a) Find all equilibrium values of y and classify each as stable or unstable. (b) What is the eventual value of y (i.e., $\lim_{t\to\infty} y(t)$) if initially (i) $y = 2$, (ii) $y = 0$, (iii) $y = -0.001$? (c) A alternate model predicts that y should satisfy the equation $dy/dt = \sin y + a$, for some positive constant a. For what values of a will this second model behave in a way which is *drastically* different from the first model?

8. Let $\mathbf{x}' = P\mathbf{x}$ be a system of two equations in two unknowns, where P is a 2×2 matrix whose coefficients are continuous functions defined for all t. Let $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ be two solutions of this system.

(a) Show that $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$ is also a solution, for c_1 and c_2 constants.

(b) Define the Wronskian $W(t)$ of $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$, and show that if $W(0) \neq 0$ then (a) can be used to solve any initial value problem $\mathbf{x}' = P\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}^0$.

(c) Give an example in which $W(0) = 1$ and $W(2) = 0$, or explain why this is impossible.

9. A mass of 1/2 slug (weighing 16 lb) is hung from the ceiling via a spring. The spring constant is 5 lb/ft and the spring has unstretched length 18 inches; the mass is subject to a linear damping force whose magnitude (in lb) is 3 times the speed of the mass (in ft/sec).

(a) How far below the ceiling is the equilibrium position of the mass?

(b) Suppose that the mass is released from a position 1 foot above its equilibrium position with a velocity of 4 ft/sec directed downward. Find the subsequent motion of the mass, and find the velocity of the mass in (b) at the first time it passes through its equilibrium position.

(c) If the mass is subject to a force $F = 2\sin t$, find the motion after a long time. (You do *not* need to know the initial position or velocity of the mass to solve this problem.) Find the amplitude and phase of this motion.

ANSWERS (not checked too carefully—be wary):

1. (a) $\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$; unstable (saddle point). (b) $(0,0)$, $(0,1)$, $(4,0)$. (c) Both $(0,1)$.

2. (a) Spiral source (unstable spiral); (b) $y(t) = (e^t/2)\begin{pmatrix} 2\cos 2t - 3\sin 2t \\ 4\cos 2t + 7\sin 2t \end{pmatrix}$.

3. (a) $c_1 e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} -1 \\ 4 \end{pmatrix}$. (b) (i) $(-1,4)$; (ii) $(1,1)$; (iii) $(-1,0)$.

4. (a) $y = a_0y_0 + a_1y_1$, $y_0 = 1 - 2x^2 + 4x^4/3 - 8x^6/15 + \cdots$, $y_1 = x - x^2 + x^5/2 - x^7/6 + \cdots$
(b) $a_0 = 3$, $a_1 = 1$. (c) $a_{2k} = (-2)^k/[(2k-1)\cdot(2k-3)\cdots 3\cdot 1]$, $a_{2k+1} = (-1)^k/k!$, $y_1 = xe^{-x^2}$.

5. (i) $\rho = \pi/2$. (ii) $\rho = 2\sqrt{2}$. (iii) $\rho = \infty$. (iv) Not ordinary point.

6. (a) Possible. (b) Impossible. See Exam 2 solutions for explanations.

7. (a) Unstable: $0, \pm 2\pi, \pm 4\pi, \ldots$; Stable: $\pm\pi, \pm 3\pi, \pm 5\pi, \ldots$. (b) $\pi, 0, -\pi$. (c) $a > 1$.

8. See book or class notes.

9. (a) 4.7 ft. (b) $u(t) = e^{-3t}(\sin t - \cos t)$.
(c) $u(t) = (12\sin t - 8\cos t)/39 = R\cos(t - \delta)$, $R = 4\sqrt{13}/13$, $\delta = 2.1588$ radians.