Lab 2: Numerical Methods

This Maple lab is closely based on earlier versions prepared by Professors R. Falk and R. Bumby of the Rutgers Mathematics department.

Introduction. In this lab, we shall use Maple's ability to approximate the solution of differential equations by numerical methods and add these solutions to the direction fields that we studied in Lab 1. This allows us to extend our understanding of the solutions of differential equations to equations for which a closed form solution is not available.

Please obtain the seed file from the web page and save it in your directory on eden, then prepare the Maple lab according to the instructions and hints in the introduction to Lab 0. Turn in only the printout of your Maple worksheet, after removing any extraneous material and any errors you have made.

Setup. As usual, the seed file begins with commands which load the required Maple packages: `with(plots):` and `with(DEtools):`.

Equation 1. *a. One Solution.* The sequence of commands to begin working with Equation 1 have already been entered into the seed file. Execute them to see how Maple can be used to plot the direction field along with the particular solution satisfying this differential equation and the initial condition $y(0) = 0$. The quantities `x1int` and `y1int` are introduced to have standard names for the *ranges* of the variables in *all* graphs in this problem. For the other exercise, corresponding commands can be constructed by copying lines to different places in a worksheet and doing some minor editing.

```
de1:= diff(y(x),x) = 3+ 4*x*y(x)
initval1a:=[y(0)=0];
x1int:=x=-1..1;y1int:=y=-3..3;
DEplot(de1, y(x), x1int, initval1a,y1int,title="Default view of equation 1");
```

The `DEplot` command has many options which you can read about in the Maple help pages. In particular, it is possible to specify the color of the solution curve by using `linecolor`, the color of the direction field by using `color`, to make the gridwork of lines in the direction field finer (or coarser) by using the `dirgrid` option, and to get a finer resolution in computing the solution with the `stepsize` option. For example, try:

```
opt1:=linecolor = black,color = green, dirgrid= [30,30],stepsize=0.1;
DEplot(de1, y(x), x1int, initval1a,y1int, opt1);
```

b. Dependence on initial condition. By varying the initial conditions, it is possible to approximate special solutions. For example, one type of solution of this equation, e.g. the one with with $y(0) = -1$, is everywhere increasing, whereas another type, e.g. the one with $y(0) = -2$ reaches a maximum value and then becomes rapidly decreasing as $x \to +\infty$. By plotting solutions with other values of $y(0)$ between -2 and -1, you can identify more precisely where this change of behavior occurs. To investigate this behavior, copy the first three command lines under part a to part (b), change the range of x to go from -1 to 3, and experiment by graphing the solution with different initial values $y(0)$. Find two initial values a and b, satisfying $-2 < a < b < -1$, which differ by 0.01 (for example, perhaps $a = -1.47$, $b = -1.46$) such that the corresponding curves are of two different types. Include in your worksheet only the graphs for these two initial conditions (together with the commands necessary to produce them) and a brief statement of what the values are.

(c) Qualitative properties. Many qualitative properties of solutions of a differential equation can be determined from the equation itself. These qualitative properties serve as a check against obvious

errors in numerical solutions. While Maple selects a fairly reliable procedure for solving initial value problems numerically, all numerical work must balance accuracy against computational effort, and sometimes a proper balance is not achieved. Simple methods to detect computed results that violate fundamental properties of solutions are the first defense against misplaced trust of computed results. For the equations in these projects, you should be able to achieve *visually accurate* results with *no noticeable computational effort*.

For this equation, the two types of solutions that we found can be partially detected by features of the equation. Conclude your study of this equation with a *discussion* of the following features of the equation related to the following properties of solutions:

1. In the quadrant where $x > 0$ and $y > 0$, solutions seem to be *steadily increasing*. How can this be proved from the differential equation?
2. Other solutions seem to have a *unique maximum point*. Such points must be *local extrema*. *Answer in text:* How do you recognize a point which is a local extremum? How can the differential equation be used to identify the possible positions of these local extrema?
3. Use of the equation in this way to identify a curve that contains all possible maximum points of the various solution curves. Report your conclusion in the worksheet.

To check your answer to question 3, a `display` could be constructed that adds the curve found there to the graphs of several different solutions to the equation, but *you are not asked to do this.*. Note that this curve would *not* be a solution; its role would only be to identify a useful feature of the solutions that should be shown accurately in the graph of a solution.

Equation 2. Consider the differential equation

$$\frac{dy}{dx} = \left(4x^2 - y^2\right)\sin y$$

with the initial condition $y(0) = -1$.

Introduce the name `de2` for this equation and use the `DEplot` command to plot a direction field and the (numerical) solution of this initial value problem for $-3 \le x \le 6$ and $-4 \le y \le 0$ with *no special options*. Although the result *will not be satisfactory*, you should *leave this graph in your worksheet* since you will discuss its features..

To get a more accurate graph, experiment with different stepsizes. As in your work on the first problem in this lab, you can vary the step size with a command like

```
opt2:=linecolor=black, color=green, dirgrid=[24,12],stepsize=0.2
```

Other stepsize choices can be made in the obvious way.

Note: DEplot uses a nominal step size of one twentieth of the total range of the x variable, in this case 0.45. This is the stepsize used for the first plot you constructed above. If you request a stepsize larger than this, Maple ignores the request and uses the default step size.

Try various choices of `stepsize` and select one that gives *what you think is an accurate graph* while not using an excessively small `stepsize`. The plot should *look smooth*, and no there should be *no delay* in computing the solution for the plot. You may also experiment with changing other options. When you have found a suitable step size, include in your final worksheet a graph using this step size, together with the instructions that you need to produce the graph. Thus the final worksheet should contain exactly two graphs based on this differential equation, and the second plot should use a small enough `stepsize` to give a smooth graph that avoids the flaws of the first graph, but not so small that you notice the time needed to compute the solution. *Include in text* a statement of the stepsize you used for the second graph.

You are now ready to *discuss* what you have learned about this equation.

(1) *Verify* that the constant functions $y = 0$ and $y = -\pi$ are solutions of the equation. Note that you don't need to know anything about how to solve an equation to recognize a solution if it is given to you. You can do this with Maple if you wish, but other methods may be easier. However you do this, include a *text summary* of how you checked this.
(2) How does this show that the solution of the given initial value problem has $0 > y(t) > -\pi$ for all t? (Answer in *text*.)
(3) How do you know that the first plot is *not* correct? What gives you confidence that your second plot accurately shows the shape of the graph of the solution of this equation?

End of Lab 2