Lab 5: A Nonlinear System

This Maple lab is based in part on earlier versions prepared by Professors R. Falk and R. Bumby of the Rutgers Mathematics department.

Introduction. In this lab we use Maple to study the phase plane of an autonomous nonlinear system of two differential equations, i.e., a system of the form

$$x'(t) = F(x, y), \qquad y'(t) = G(x, y). \tag{1}$$

The distinguishing feature of an autonomous system is that the expressions defining the functions F and G do not contain the independent variable t. This allows many properties of the solutions to be studied using the curves, called *trajectories*, that show the path in the xy plane followed by the solutions. (It is an easy exercise to show that if an initial condition is on a trajectory, then the whole solution follows that trajectory). The Maple command `DEplot` may be used to draw trajectories and direction fields for such systems.

Please obtain the seed file from the web page and save it in your directory on eden, then prepare the Maple lab according to the instructions and hints in the introduction to Lab 0. Turn in only the printout of your Maple worksheet, after removing any extraneous material and any errors you have made.

0. Setup. As usual, the seed file begins with commands which load the required Maple packages: `with(plots):`, `with(DEtools):`, and `LinearAlgebra`.

1. An autonomous nonlinear system: a. The equation and its critical points..

We will study the nonlinear system

$$\begin{aligned} x' &= 4x + xy - x^2 = x(4 + y - x) \\ y' &= 6x + 6y - xy - x^2 = (6 - x)(x + y). \end{aligned} \tag{2}$$

The equilibrium solutions are $[x = 0, y = 0]$, $[x = 2, y = -2]$, and $[x = 6, y = 2]$. Maple can obtain these by using the instructions

```
F:=2*y - 2*x + x*y - x^2;
G:=4*y + 4*x - x*y - x^2;
eqpts:=solve(F,G,x,y);
```

which are included in the seed file. Also included are instructions to define the differential equations:

```
dex:=diff(x(t),t)=eval(F,x=x(t),y=y(t));
dey:=diff(y(t),t)=eval(G,x=x(t),y=y(t));
```

For later convenience we have defined F and G to depend on the variables x and y, but in the differential equations we must write these variables as $x(t)$ and $y(t)$; the `eval` command makes this substitution.

b. The Direction field and nullclines. The next instructions establish an appropriate range for the independent variable and a useful viewing window, and construct a plot of the direction field of this system. The plot consists of small arrows pointing the way of the trajectories in the square $-4 \le x \le 8$, $-6 \le y \le 6$.

```
trange := -8..8:  window:=x=-4..8,y=-6..6:
df:=DEplot([dex,dey],[x(t),y(t)], trange, window,color=GREEN):
```

(Note the colon at the end of the instruction defining the plot, suppresses output of the plot structure). The `color` option is used to give a better view of the *nullclines* and solution curves to be added later. If you like, you can test different values of the option `dirgrid`, as in Lab 2, and customize the value for the plots in this project. A value should be chosen that allows individual arrows to be identified while including a sufficient number of arrows. To show this plot, you need only enter its name `df1;` after defining it. The result *should not be left in the worksheet*, since we will include this as part of a `display` command in the next subsection.

b. Nullclines. The points where the direction field is *horizontal* (characterized by $dy/dt = 0$) or *vertical* (characterized by $dx/dt = 0$) form curves called *nullclines*. In many cases, these curves provide useful information about the behavior of trajectories without the excessive detail of a direction field.

For the system (2), the factored form of the expressions for dx/dt and dy/dt allows a simple description of the nullclines. In particular, the slope field is horizontal along the curves $x = 6$ and $y = -x$. Since $x = 6$ is a vertical line, it is best to plot these curves parametrically; such a plot can be constructed using

```
dh:=plot([[6,t,t=trange],[t,-t,t=trange]],window,color=CORAL):
```

Construct a similar instruction to produce a plot `dv` showing where the slope field is vertical; to be definite, use `color=BLUE`. Now obtain a `display` (including a title) combining the slope field and both sets of nullclines:

```
display(df,dh,dv,title="Slope field and nullclines:  Nonlinear equation");
```

The equilibrium points should be seen as points lying on one nullcline of each color. Note that the nullclines usually *cut across the arrows* in the slope field since they are *not* solutions of the equation (except in rare cases).

Discussion: For each of the three equilibrium points found in part **a**, give explicitly the equations of the vertical and the horizontal nullclines which cross at that point.

c. Trajectories. To study the behavior of the system and the nature and stability of equilibrium points, it is also useful to construct a plot of the phase plane which shows several trajectories as well as the direction field. In Lab 4, we did this by finding the *exact* solution to the equation. Since this is in general not be possible for nonlinear equations, we use here the *numerical methods* that are part of the `DEplot` command. To produce each trajectory one specifies an initial condition—i.e., a point in the phase plane—and asks Maple to produce the trajectory through that point; to get a significant part of the full trajectory one solves over a range of the independent variable, t, which includes both positive and negative values.

To obtain a good choices of initial conditions, leading to a set of trajectories which illustrate the important features of the phase plane, requires some trial and error. The seed file contains a list of six initial conditions, together with a command to produce a plot of the phase plane containing the direction field and the corresponding six trajectories:

```
inits:=[[x(0)=1,y(0)=1],[x(0)=2,y(0)=-1],[x(0)=3,y(0)=-2],
     [x(0)=1,y(0)=-4],[x(0)=-1,y(0)=2],[x(0)=2,y(0)=-3]]:
DEplot([dex,dey],[x(t),y(t)],t=trange,inits,window,color=GREEN,
     linecolor=[RED,BLUE,BROWN,PLUM,CORAL,BLACK],thickness=2,stepsize=0.005,
     title="Trajectories:  Nonlinear equation");
```

The choice of stepsize here seems to work well, but you can experiment with other possible choices. Include the resulting plot in your final worksheet.

Discussion: By examining the phase plane plot, you should be able to decide the *type* of each of the three critical points: stable or unstable node, saddle point, or stable or unstable spiral. Give your conclusions for each point in the *discussion* section,

Discussion: In the second *discussion* section you should describe any interesting properties of each of the six trajectories shown. In particular, *for each one*, discuss whether or not that trajectory appears to approach a critical point—and if so, which one—as $t \to \infty$ and as $t \to -\infty$. In some cases you will not be able to determine this from your graph, and it is perfectly acceptable to say so. In your discussion you should identify the various trajectories by color.

d. Linearization. The type and stability of the critical points can be determined by examining the eigenvalues of the corresponding linear system. The matrix of the linearization of the system (1) at the critical point (x_c, y_c) is

$$A := \begin{pmatrix} F_x(x_c, y_c) & F_y(x_c, y_c) \\ G_x(x_c, y_c) & G_y(x_c, y_c) \end{pmatrix}.$$

Maple can find this matrix in a two step process, first constructing the matrix of partial derivatives (which must be done only once), then evaluating it at the critical point (done separately for each point). Instructions for finding A at the critical point in the fourth quadrant, and its eigenvalues, are in the seed file:

```
A := Matrix([[diff(F,x),diff(F,y)],[diff(G,x),diff(G,y)]]);
A1 := eval(A,x=2,y=-2);
Eigenvalues(A1);
```

Use Maple to find the matrices $A2$ and $A3$, and their eigenvalues, corresponding to the critical points at the origin and in the first quadrant, respectively.

Discussion: In a *discussion section*, for each critical point, describe what the eigenvalues you found imply about the type of that point, and compare the answer with the one you found in part **c**. (The answers should agree!)

The phase plane near the critical points. In sections **e**, **f**, and **g** we sketch the phase plane for the true system, and for its linearization, near each of the three critical points. We use a square window, two units by two units, centered at the critical point, and as initial conditions the four points $(x_c \pm 0.5, y_c \pm 0.5)$. In sections **e** and **f** we also introduce one extra trajectory which is a "special trajectory"—that is, a straight-line trajectory—for the linearized system.

Discussion: In each section you should produce two plots. Include also a discussion comparing the plots for the true and linearized systems: what similarities do you see? What differences? In sections **e** and **f** discuss also the extra trajectory, as described below.,

e. The phase plane near $(2, -2)$. For this case the seed file contains the commands needed to produce the two plots. The commands needed to graph the phase plane of the nonlinear system near $(2, -2)$ are

```
trange1 := -3..3:  window1 := x=1..3,y=-3..-1:
inits1:=[[x(0)=2.5,y(0)=-1.5],[x(0)=1.5,y(0)=-1.5],[x(0)=1.5,y(0)=-2.5],
   [x(0)=2.5,y(0)=-2.5],[x(0)=2+2.0/3.0,y(0)=-2+(3.0-sqrt(17.0))/3.0]]:
DEplot([dex,dey],[x(t),y(t)],t=trange1,inits1, window1,color=GREEN,
   linecolor=[RED,BLUE,CYAN,PLUM,BLACK],thickness=2,stepsize=0.002
   title="Phase plane near (2,-2):  nonlinear system");
```

For the linearized system the commands are

```
F1:=-2*u+2*v;
G1:=4*u+4*v;
dex1:=diff(x(t),t)=eval(F1,u=x(t)-2,v=y(t)+2);
dey1:=diff(y(t),t)=eval(G1,u=x(t)-2,v=y(t)+2);
DEplot([dex1,dey1],[x(t),y(t)],t=trange1,inits1, window1,color=GREEN,
   linecolor=[RED,BLUE,CYAN,PLUM,BLACK],thickness=2,stepsize=0.002,
   title="Phase plane near (2,-2):  linearized system");
```

Note that here:

(i) The functions $F1$ and $G1$ are obtained from the matrix $A1$ found in part **d**.

(ii) In defining the linear differential equations we make the substitutions $u = x - x_c = x - 2$ and $v = y - y_c = y + 2$, so that the resulting phase portrait is centered (in the xy plane) at the point $(2, -2)$. This provides for easy comparison with the nonlinear case.

Note also that in addition to the four initial conditions $(2\pm0.5, -2\pm0.5)$ we have an initial condition $(2, -2) + (\xi_1, \xi_2)/3$, where $\xi = (2, 3 - \sqrt{17})$ is one of the eigenvectors for the linear problem (found using the command `Eigenvectors(A1)`). The corresponding trajectory is a straight line in the linear system, but not in the nonlinear system.

Discussion: In your *discussion*, explain why this (black) trajectory looks so different in the two plots, while the other trajectories are similar.

f. The phase plane near $(0, 0)$**.** Modify the above commands to produce plots of the phase plane for the nonlinear and linear systems near the origin. In addition to the four initial conditions $(\pm0.5, \pm0.5)$, include also the initial condition $(0, 0.5)$. This produces a straight line trajectory in the linear system, but, in contrast to what happened in part **e**, also a straight line trajectory in the nonlinear system.

Discussion: In your *discussion*, explain why the black trajectory is straight even in the nonlinear system.

g. The phase plane near $(6, 2)$**.** Modify the above commands to produce plots of the phase plane for the nonlinear and linear systems near $(6, 2)$. There are no straight line trajectories to consider in this case..

End of Lab 5