SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

We want to discuss the solutions of the *first order system of ordinary differential equations*

$$\mathbf{x}' = \mathbf{P}\mathbf{x} + \mathbf{f}, \tag{1}$$

where $\mathbf{x}$ and $\mathbf{f}$ are vector functions of t, with f known and $\mathbf{x}$ unknown—$\mathbf{x}$ is the *dependent variable*—and $\mathbf{P}$ is a known matrix function of t:

$$\mathbf{x} = \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{f} = \mathbf{f}(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix} \quad \mathbf{P} = \mathbf{P}(t) = \begin{pmatrix} p_{11}(t) & \dots & p_{1n}(t) \\ \vdots & & \vdots \\ p_{n1}(t) & \dots & p_{nn}(t) \end{pmatrix}.$$

Throughout we assume that we are looking for a solution $\mathbf{x}(t)$ defined on some interval I, and that all the functions $p_{ij}(t)$ and $f_i(t)$ are continuous functions of t for t in the interval I.

CASE I: The homogeneous case: $\mathbf{f} = 0$.

In this case we want to solve the *homogeneous system*

$$\mathbf{x}' = \mathbf{P}\mathbf{x}. \tag{2}$$

There are two important principles:

PRINCIPLE OF SUPERPOSITION: *If $\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$, ..., $\mathbf{x}^{(k)}(t)$ are all solutions of (2), and c_1, c_2, ..., c_k are constants, then*

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \cdots + c_k\mathbf{x}^{(k)}(t) \tag{3}$$

is also a solution of (2).

The principle of superposition tells us how to build new solutions from solutions we already have; $\mathbf{x}(t)$ in (3) is called a *superposition* or *linear combination* of the solutions $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$. The next principle tells us how to build all solutions.

GENERAL SOLUTION: *Suppose that $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, ..., $\mathbf{x}^{(n)}$ are n solutions of (2), linearly independent on the interval I. Then* every *solution $\mathbf{x}$ of (2) is of the form*

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t) \tag{4}$$

for some constants $c_1, c_2, \ldots, c_n$.

It is important to realize that there are three conditions here on the vector functions $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ which are necessary to guarantee that the general solution has the form (4). First, the $\mathbf{x}^{(i)}$ must themselves be solutions of (2). Second, there must be exactly n solutions: *the same number of solutions as the number of components in the vectors, i.e., as the number of different first order equations represented in vector form in (2).* Third, these n solutions must be *linearly independent.*

The easiest way to determine linear independence of solutions is usually to use the *Wronskian*:

LINEAR INDEPENDENCE OF SOLUTIONS AND THE WRONSKIAN: *Suppose that we have precisely* n *functions* $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t)$, *all of which are solutions of the homogeneous equation (2). Let the* Wronskian $W(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)})(t)$ *be the determinant of the matrix whose columns are these vectors. Then these solutions are linearly independent on* I *if and only if the Wronskian is nonzero on* I.

However, there is a little more to say about linear independence. When we are talking about vectors and vector functions, this term can be used in two different ways.

First, a collection of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ (constant vectors—not functions of t) is *linearly independent* if the vectors have the property that whenever a linear combination $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$ is the zero vector, then necessarily all the coefficients $c_1, \ldots, c_k$ are zero.

Linear independence of vectors:

$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = 0 \qquad \Rightarrow \quad c_1 = \cdots = c_k = 0.$$

(A set of vectors which is not linearly independent is called *linearly dependent.*) If there are n vectors—the same number as the number of components—then the simplest way to check linear independence is to form the square matrix whose columns are the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$: the determinant of this matrix is non-zero if and only if the vectors are linearly independent.

Second, a collection of vector functions $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(k)}(t)$, all defined on some interval I, is *linearly independent on* I if they have the property that whenever a linear combination

$c_1\mathbf{x}^{(1)}(t)+\cdots+c_k\mathbf{x}^{(k)}(t)$ is the zero vector for all t in I, then necessarily all the coefficients $c_1, \ldots, c_k$ are zero.

Linear independence of vector functions:

$$c_1\mathbf{x}^{(1)}(t)+\cdots+c_k\mathbf{x}^{(k)}(t)=0 \quad \text{for all } t \text{ in } I \qquad \Rightarrow \qquad c_1=\cdots=c_k=0.$$

In general, the relation between these two notions of linear independence may be complicated. The Wronskian of vector functions on an interval may be zero at some points and non-zero at others, corresponding to the linear dependence or independence of the (vector) function values at those points. However, there is one case when it is simple.

LINEAR INDEPENDENCE OF SOLUTIONS: *Suppose that we have precisely n functions $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t)$, all of which are solutions of the homogeneous equation (2). Then* either

(i) $W(t) = 0$ for all *t in I, in which case the vector functions $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t)$ are linearly dependent on I and, for any fixed point t_0 in I, the (constant) vectors $\mathbf{x}^{(1)}(t_0), \ldots, \mathbf{x}^{(n)}(t_0)$ are linearly dependent, or*

(ii) $W(t) \neq 0$ for all *t in I, in which case the vector functions $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t)$ are linearly independent on I and, for any fixed point t_0 in I, the (constant) vectors $\mathbf{x}^{(1)}(t_0), \ldots, \mathbf{x}^{(n)}(t_0)$ are linearly independent.*

CASE II: The inhomogeneous case: $\mathbf{f} \neq 0$.

In this case we want to solve the *inhomogeneous system* (1). The principle here is familiar:

GENERAL SOLUTION OF INHOMOGENEOUS EQUATION: *Suppose that $\mathbf{v}(t)$ is some (particular) solution of the inhomogeneous equation (1) and that $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t)$ are n linearly independent solutions of the corresponding homogeneous equation (2). Then every solution of the inhomogeneous equation is of the form*

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t) + \mathbf{v}(t).$$