Mathematics 244: Lab 2 Direction Fields and Numerical Methods

In this lab, we shall use Maple's ability to plot direction fields and approximate the solution of differential equations by numerical methods to understand the solutions of differential equations.

Please turn in only the printout of your Maple worksheet. Include explicit answers to all questions asked, using the **text** feature of Maple to insert them in the worksheet. Remove from the worksheet any extraneous material and any errors you have made.

Before you start up Maple, first copy the file lab2.mws into your directory from the Web page of the course. After you start up Maple, import this file into Maple by choosing **Open** from the **File** menu as described in the handout **Instructions for Use of Maple in Mathematics 244**. You will be asked to execute the Maple commands that have been placed in this worksheet and to add some of your own.

1a. In Lab 1, we saw how the direction field of a differential equation could be obtained by use of the command `dfieldplot`. In cases where the exact solution of an initial value problem can be obtained, this solution may be plotted together with the direction field by combining the `dfieldplot`, `plot` and `display` commands. A more direct method, which gives results even in cases in which an exact solution of the differential equation can not be found, is to use the command `DEplot`. This command computes a numerical solution to a differential equation and then plots the resulting solution curve together with the direction field of the differential equation.

The sequence of commands described in Problem 1a have already been entered into the Maple worksheet lab2.mws. Execute them to see how Maple can be used to plot the direction field of the equation $y' = 1 + 2xy$ and then the direction field along with the particular solution satisfying this differential equation and the initial condition $y(-1) = -2$.

```
with(plots): with(DEtools):
de:= diff(y(x),x) = 1 + 2*x*y(x)
dfieldplot(de, y(x), x=-3..3, y=-3..3);
DEplot(de, y(x), x=-3..3, y=-3..3);
initval:={[y(-1)=-2]};
DEplot(de, y(x), x=-3..3, initval,y=-3..3);
```

The `DEplot` command has many options which you can read about by typing `?DEplot`. In particular, it is possible to specify the color of the solution curve by using `linecolor`, the color of the direction field by using `color`, and to make the gridwork of lines in the direction field finer by using the `dirgrid` option. For example, try:

```
DEplot(de, y(x), x=-3..3, initval, y=-3..3, linecolor = black,
color = green, dirgrid= [30,30]);
```

It is also possible to plot several solution curves on the same graph. For example, type

```
initval:= {[y(-3)=-2],[y(-1)=-2],[y(1)=-2]};
DEplot(de, y(x), x=-3..3, initval, y=-3..3);
```

1b. By trial and error, find a number x_0 with $-1 \leq x_0 \leq 0$ such that the solution curve through $[x_0, -2]$ passes through $[2, 1]$. To determine this value of x_0 more precisely, use `dirgrid=[30,30]` and change the x and y ranges in `DEplot`. Check your result by plotting the solution curves for the initial values $[x_0, -2]$ and $[2, 1]$ on the same graph for the value of x_0 that you found.

1c. Does there exist a number x_1 with the property that the solution curve through the point $[x_1, 2]$ also passes through the point $[1, -2]$? If so, find it, and if not, use the plot of the direction field to explain why.

1d. The solution $y(x)$ of the differential equation $y' = 1 + 2xy$ satisfies at each x, $y'(x) = 1+2xy(x)$. Therefore, at points where $y'(x) = 0$, $y(x) = -1/(2x)$. On the same set of axes, plot the function $y = -1/(2x)$, the direction field for the equation $y' = 1 + 2xy$, and the solutions of this equation corresponding to the initial conditions $y(-1) = 2$, $y(-1) = -1$, and $y(0) = -2$ over the interval $-3 \leq x \leq 3$. Note that to place these on the same graph, plot the function $y = -1/(2x)$ and give the output a name (say `q1`) and plot the direction field and solutions of the differential equation giving the output the name `q2`. When giving a name to the output of a plot, you can avoid unwanted output by ending the command with a colon, rather than a semicolon. The command `display(q1,q2);` will then display both plots together.

1e. What do you notice about (i) the direction of the arrows along the curve $y = -1/(2x)$ and (ii) the points of intersection of this curve with the solution curves of the differential equation?

1f. By hand computation, obtain an expression for d^2y/dx^2 in terms of x and y by differentiating the differential equation $dy/dx = 1 + 2xy$ and then using the differential equation to eliminate dy/dx. By hand computation, now determine y as a function of x which solves the equation $d^2y/dx^2 = 0$. Obtain a plot on the same set of axes of the function y you just obtained, the direction field for the equation $y' = 1 + 2xy$, and the solutions of this equation corresponding to the initial conditions $y(-1) = 2$, $y(-1) = -1$, and $y(-1) = -2$ over the interval $-3 \leq x \leq 3$. This can be done using Maple in a manner similar to part (d).

1g. What do you notice about the points of intersection of these curves?

2. Consider the differential equation $x'(t) = (x-1)^2(x+1)$. The equilibrium solutions of this equation are easily seen to be $x = 1$ and $x = -1$.

2a. Plot the direction field for this equation over the region $-3 \leq t \leq 3$, $-3 \leq x \leq 3$.

2b. By viewing the plot of the direction field, state the values that solutions of the equation can approach as $t \to \infty$.

2c. For each of the values in part (b), state the region in the (x, t) plane such that if a solution begins in that region, it will approach the particular value.

2d. For each of the values in part (b), plot a solution curve that tends toward that value.

All these solution curves should be placed on the same plot. This can be done using `DEplot` in a manner similar to Problem 1.

3. Consider the differential equation

$$\frac{dy}{dt} = y \ln(1/y)(2 - y)$$

which is sometimes used to model population growth.

3a. To get a rough idea about the behavior of this model, plot on the same set of axes the direction field and solutions corresponding to the initial conditions $y(0) = 0.2$, $y(0) = 0.8$, $y(0) = 1$, $y(0) = 1.2$, $y(0) = 1.8$, $y(0) = 2$, $y(0) = 2.2$. Use the ranges $0 \le t \le 5$ and $0 \le y \le 2.5$.

3b. Based on the graph of part (a), determine the equilibrium solutions and state which are stable and which unstable.

3c. Although `DEplot` can be used to obtain qualitative information about the solution of a differential equation by plotting its direction field and some of the solution curves, it does not give quantitative information. To obtain quantitative information about the solution of an initial value problem for which we have no exact solution, we use the `numeric` option of the `dsolve` command. To use this command to obtain a numerical solution of the differential equation given above (named `de`), with the initial condition $y(0) = 1.2$, giving the result the name `sol`, type

```
sol:=dsolve({de,y(0)=1.2},y(t),numeric);
```

The result of this Maple command is a Maple procedure, which acts like a function. Evaluate the numerical solution at $t = 2$ by typing `sol(2);`

3d. Find the time t, correct to one decimal place, at which $y(t) = 1.5$. This can be done by using the numerical solution obtained above in `sol` and trial and error to evaluate it at various points. The plot in part (a) should give you a rough idea of the correct value.

3e. Note that when you type `sol(2)`, Maple does not just return the value, but instead returns a list of equations. The practical effect of this is that one cannot get a graph of the solution by just typing `plot(sol,t=0..5);`. Try this to see that the plot Maple gives is clearly wrong. Instead, the numerical solution obtained by using `dsolve[numeric]` can by plotted by using the Maple command `odeplot`. In this case, type `odeplot(sol,[t,y(t)],0..5);` to obtain a plot.