

Mathematics 244: Lab 3 Second Order Differential Equations

In this lab, we use Maple to examine differential equations modeling forced and unforced oscillations. In particular, we shall consider linear and nonlinear models of a pendulum and also explore the phenomena of beats and resonance.

Please turn in only the printout of your Maple worksheet. Include explicit answers to all questions asked, using the **text** feature of Maple to insert them in the worksheet. Remove from the worksheet any extraneous material and any errors you have made.

Before you start up Maple, first copy the file lab3.mws into your directory from the Web page of the course. After you start up Maple, import this file into Maple by choosing **Open** from the **File** menu as described in the handout **Instructions for Use of Maple in Mathematics 244**. You will be asked to execute the Maple commands that have been placed in this worksheet and to add some of your own.

1. Consider the motion of a pendulum, which consists of a mass m attached to one end of a rigid rod of length L . The other end of the rod is fixed at a point O and the rod is free to rotate about O . The position of the pendulum at time t is described by the angle $\theta(t)$ between the rod and the downward vertical direction, with the counterclockwise direction taken as positive (see the figure on page 476 of the text). The differential equations governing the motion of the pendulum are

$$\frac{d^2\theta}{dt^2} + \frac{c}{mL} \frac{d\theta}{dt} + \frac{g}{L} \sin \theta = 0$$

in the case where a damping force modeling air resistance is included, and

$$\frac{d^2\theta}{dt^2} + \frac{g}{L} \sin \theta = 0,$$

if the damping force is neglected. The constant g is the acceleration due to gravity, which we take as $9.81m/sec^2$. Since these are nonlinear differential equations, they are often studied by considering simpler linear models obtained by using the approximation $\sin \theta \approx \theta$, which is valid when θ is small. (Note that θ is just the first term of the Taylor series for $\sin \theta$ about $\theta = 0$.) In that case, we have the linear pendulum equation

$$\frac{d^2\theta}{dt^2} + \frac{c}{mL} \frac{d\theta}{dt} + \frac{g}{L} \theta = 0.$$

In this problem, we study the differences between the linear and nonlinear models and the effect of damping on the solution. To simplify the equations, consider a pendulum of length 1.09 meters and values of c and m so that $\gamma = c/(mL)$ takes on the values 1, 6, and 9. Consider the initial conditions $\theta(0) = 1$, $\theta'(0) = 4$.

The **dsolve** command can be used to compute the exact solution of the linear models. Its use is not exactly the same as for first order equations, since we now have two initial

conditions. To solve the linear equation without damping and plot the solution, execute the following commands.

```
de1:=diff(x(t),t$2) + 9*x(t)= 0;
ivp1:={de1,x(0)=1, D(x)(0)=4};
soln1:=dsolve(ivp1,x(t));
plot(rhs(soln1),t=0..10);
```

1b. Use `dsolve` (in a manner similar to part (a)) to obtain solutions to the linear damped equations for the three cases $\gamma = 1, 6, 9$ and plot the solutions over the interval $0 \leq t \leq 10$.

1c. For the nonlinear equations, no exact solution in elementary functions can be found and thus the standard use of the `dsolve` command does not give a useful result. However, the `numeric` option of the `dsolve` command can be used to compute numerical approximations to the solutions of the nonlinear models. To solve the nonlinear equation without damping and plot the solution, execute the following commands.

```
de5:=diff(x(t),t$2) + 9*sin(x(t))= 0;
ivp5:={de5,x(0)=1, D(x)(0)=4};
soln5:=dsolve(ivp5,x(t),numeric);
odeplot(soln5,[t,x(t)],0..10);
```

Also note what happens if you execute:

```
dsolve(ivp5,x(t));
```

If you get a solution, you should examine it to see how you would discover its properties.

To compare the solution of the linear and nonlinear equations, execute the following:

```
p1:=plot(rhs(soln1),t=0..10,color=blue):
q5:=odeplot(soln5,[t,x(t)],0..10):
display({p1,q5});
```

Note that the qualitative behavior is similar, but the trajectories are different.

1d. Use the `numeric` option of `dsolve` to obtain solutions to the nonlinear damped equations for the three cases $\gamma = 1, 6, 9$ and plot the solutions over the interval $0 \leq t \leq 10$. Then for each choice of γ , plot on the same set of axes the solution of the linear and nonlinear equation for that value of γ .

2. Consider the initial value problem

$$x'' + 25x = \sin(\omega t), \quad x(0) = 0, \quad x'(0) = 0.$$

2a. Find a value of ω so that the solution of the above initial value problem displays resonance. Use `dsolve` to find the exact solution and then `plot` to plot the solution.

2b. Find a value of ω so that the solution of the above initial value problem displays beats. Use `dsolve` to find the exact solution and then `plot` to plot the solution.