Mathematics 244: Lab 2 Direction Fields and Numerical Methods

In this lab, we shall use Maple's ability to plot direction fields and approximate the solution of differential equations by numerical methods to understand the solutions of differential equations.

Please turn in only the printout of your Maple worksheet. Include explicit answers to all questions asked, using the **text** feature of Maple to insert them in the worksheet. Remove from the worksheet any extraneous material and any errors you have made.

Before you start up Maple, copy the seed file into your directory from the Web page of the course. After you start up Maple, import this file into Maple by choosing **Open** from the **File** menu as described in the handout **Instructions for Use of Maple in Mathematics 244**. You will be asked to execute the Maple commands that have been placed in this worksheet and to add some of your own.

As usual, the seed file begins by loading the `plots` and `DEtools` libraries. Several plots are produced in these exercise. **Be sure to supply tiles for each plot that is shown in your worksheet.** (Titles are not shown in this description, but plot command in the seed file include titles.)

1a. In Lab 1, we saw how the direction field of a differential equation could be obtained by use of the command `DEplot`. In cases where the exact solution of an initial value problem can be obtained, this solution may be plotted together with the direction field by combining the `DEplot`, `plot` and `display` commands. A more direct method gives initial conditions to the `DEplot` command. In this case, Maple computes a numerical solution to a differential equation and then plots the resulting solution curve together with the direction field of the differential equation.

The sequence of commands described in Problem 1a have already been entered into the seed file. Execute them to see how Maple can be used to plot the direction field along with the particular solution satisfying this differential equation and the initial condition $y(-1) = -2$.

```
de1:= diff(y(x),x) = 1 + 2*x*y(x)
initval1a:={[y(0)=0]};
DEplot(de1, y(x), x=-3..3, initval1a,y=-3..3);
```

The `DEplot` command has many options which you can read about in the Maple help pages. In particular, it is possible to specify the color of the solution curve by using `linecolor`, the color of the direction field by using `color`, to make the gridwork of lines in the direction field finer by using the `dirgrid` option, and to get a finer resolution in computing the solution with the `stepsize` option. For example, try:

```
opt1:=linecolor = black,color = green, dirgrid= [30,30],stepsize=0.1
DEplot(de1, y(x), x=-3..3, initval1a, y=-3..3, opt1);
```

1b. It is also possible to plot several solution curves on the same graph. For example, type

```
initval1b:= {[y(0)=-2],[y(0)=-1],[y(0)=0]};
DEplot(de1, y(x), x=-3..3, initval1b, y=-3..3,opt1);
```

1c. By varying the initial conditions, it is possible to approximate special solutions. For example, the solution of this equation with $y(0) = 0$ is everywhere increasing, but the solution with $y(0) = -1$ reaches a maximum value and then becomes rapidly decreasing. By plotting solutions with other values of $y(0)$ between 0 and -1, you can identify more precisely where this change of behavior occurs. **Produce a graph** with values of $y(0)$ separated by 0.1 containing **two solutions** of each type.

1d. The solution $y(x)$ of the differential equation $y' = 1 + 2xy$ satisfies at each x, $y'(x) = 1 + 2xy(x)$. Therefore, at points where $y'(x) = 0$, $y(x) = -1/(2x)$. On the same set of axes, plot the function $y = -1/(2x)$, the direction field for the equation $y' = 1+2xy$, and the solutions of this equation corresponding to the initial conditions $y(-3) = 1/6$, $y(-3/4) = 2/3$, $y(-1/4) = 2$, $y(1/4) = -2$, $y(3/4) = -2/3$, $y(3) = -1/6$ over the interval $-3 \le x \le 3$. Note that to place these on the same graph, plot the function $y = -1/(2x)$ in a distinctive color and give the output a name (say `gr1n`) and plot the direction field and solutions of the differential equation using the standard options for this equation and give the output the name `gr1s`. When giving a name to the output of a plot, you can avoid unwanted output by ending the command with a colon, rather than a semicolon. The command `display(gr1n,gr1s);` will then display both plots together. You should add a title to the display command but not to the commands that produce the individual plots that combines.

1e. Why do the values of $y(0)$ for the solutions meeting the portion of $y = -1/(2x)$ in the fourth quadrant form an interval consisting of all values less than the value investigated in 1b? What behavior as $x \to +\infty$ do you expect for the curve with $y(0)$ equal to this value?

What do you notice about (i) the direction of the arrows along the curve $y = -1/(2x)$ and (ii) the points of intersection of this curve with the solution curves of the differential equation?

2. Consider the differential equation

$$\frac{dy}{dt} = \left(t^2 - y^2\right) \sin y$$

with the initial condition $y(0) = -1$ (This initial value problem appear as Problem 6 in Section 8.3 of the textbook).

2a. Introduce the name `de2` for this equation and use the `DEplot` command to plot a direction field and the (numerical) solution of this initial value problem for $-4 \le t \le 6$ and $-4 \le y \le 0$ with **no special options**. The result will not be satisfactory, and you should not try for a better graph.

2b. Now introduce the options

```
opt2:=linecolor = black,color = green, dirgrid= [24,12],stepsize=0.2
```

and add `opt2` to the `DEplot` command to get an improved plot. Your worksheet should contain both the original and improved plots, suitably titled.

2c. How do you **know** that the the solution of the initial value problem has $y(t) > -\pi$ for all t? How does this help recognize that the first graph is inaccurate?

3. Maple help claims that the `DEplot` command uses the classical fourth order Runge-Kutta method (as described in Section 8.3) for producing numerical solutions to initial value problems. For the interval $[-4, 6]$, the standard step size is 0.5, and our option improved it to 0.2. Under normal circumstances, this reduces error by a factor of about 40, but that additional accuracy in this case was enough to avoid a major disruption.

It is possible to use other methods for the numerical work. While Euler's method is **less accurate**, its simplicity will allow a clearer picture of the cause of the unsatisfactory behavior of the solution.

3a. To use Euler's method (with a step size of 0.5), introduce the option

```
opt3:=method=classical[foreuler];
```

Then give the `DEplot` command for the initial value problem described in 2, with this option (and no other options).

3b. Enter

```
opt3a:=opt2,opt3;
```

in order to combine the refinements of 2b with the use of Euler's method. Plot the solution with the combined options.

3c. What does the graph using Euler's method reveal about the cause of the inaccuracy with a step size of 0.5? In particular, can you identify why it is so large? Again, the smaller step size seems to avoid drastic errors. You should consider this in your discussion.

4. A graph that is 99% accurate (i.e., accurate to 2 significant figures) will look perfect, but you would probably not consider a result to have been **computed** unless you have the 12 significant figures that you have come to expect from your calculator. Maple is also able to compute values to this accuracy. The usual behavior of the `numeric` option in `dsolve` is to construct a procedure for computing the function at desired points. Once defined, this procedure can be used later to find particular values of the solution.

The seed file contains instructions illustrating the use of the fourth order Runge-Kutta method with step sizes 0.1, 0.05 and 0.025 to approximate $y(1)$ for the initial value problem introduced in 2. Although the exact value of $y(1)$ is not known, the fact that this is a fourth order method can be seen by considering the ratio of differences between values obtained with three different step sizes. This is because the distance to the true value is very close to a fixed multiple of the difference between the computed values. For a fourth order method, halving the step divides the error by 16. The dependence isn't that precise, but you should see a ratio of roughly that size.

Divide the step size in half once more and check the ratio of differences between consecutive approximations of step size 0.05, 0.025 and 0.0125.

Discuss the results. If you do not get the result you expect suggest a reason.

The instructions in this section are easily modified for use with the other classical methods. You may find such a modification useful in your discussion.

The help page for `dsolve/numeric/IVP` gives information about the methods available for producing different numerical values of the solution at several points. The standard accuracy of the recommended adaptive methods appears to be only 8 significant figures, but it is easy to set the options that request additional accuracy. This will not be explored here, but you may find it useful if you ever need a numerical solution to a differential equation.