## Mathematics 244: Lab 5 Trajectories in the Phase Plane

In this lab, we shall use Maple to study the qualitative properties of autonomous systems of two differential equations.

Please turn in only the printout of your Maple worksheet. Include explicit answers to all questions asked, using the **text** feature of Maple to insert them in the worksheet. Remove from the worksheet any extraneous material and any errors you have made.

Before you start up Maple, copy the seed file into your directory from the Web page of the course.

Each section of this lab deals with a system of two **autonomous** differential equations, i.e., a system of the form

$$x'(t) = F(x, y), \qquad y'(t) = G(x, y).$$

Note that the distinguishing feature of an autonomous system is that the functions $F$ and $G$ are independent of the variable $t$. This allows many properties of the solutions to be studied using the curves, called **trajectories**, that show the path in the $xy$ plane followed by the solutions (It is an easy exercise to show that the if an initial condition is on a trajectory, then the whole solution follows that trajectory). The Maple command `DEplot` may be used to draw trajectories and direction fields for such systems.

1a. Consider the linear system: $x' = x + 2y, \qquad y' = -5x - 2y.$

To study the stability of the equilibrium solution $[0, 0]$, execute the following commands (which appear in the worksheet lab5.mws) to enter this system into Maple and plot its direction field. This will draw small arrows pointing the way of the trajectories in the square $-6 \le x \le 6$, $-6 \le y \le 6$.

```
with(plots): with(DEtools): with(LinearAlgebra):
window:=x=-6..6,y=-6..6;
trange:=-5..5;
de1:= diff(x(t),t) = x(t) +2*y(t);
de2:= diff(y(t),t) = -5*x(t) - 2*y(t);
DEplot([de1,de2],[x(t),y(t)], trange, window);
```

1b. To study the stability of equilibrium solutions, it is also useful to draw the direction field together with several trajectories. To draw the direction field together with the trajectories through the points $[-1, 2]$, $[.5, .7]$, $[0.1, -1]$, and $[-2, -3]$, execute the commands

```
inits:={[x(0)=-1,y(0)=2],[x(0)=0.5,y(0)=0.7],[x(0)=0.1,y(0)=-1],
[x(0)=-2,y(0)=-3]};
DEplot([de1,de2],[x(t),y(t)], trange, inits, window);
DEplot([de1,de2],[x(t),y(t)], trange, inits, window, stepsize=.1);
```

Note the difference between the last two plots. In the first one, the trajectories do not follow the arrows very closely, while in the second one, they do. When `DEplot` is used, the plot is generated from approximations to the solution at points at a distance `stepsize` apart. If this number is too large, the plot will be inaccurate. The use of the option `stepsize` allows us to change the default value (in this case .5) to a smaller value (in this case .1) to obtain a more accurate plot.

To draw the trajectories without the arrows, execute the command

```
DEplot([de1,de2],[x(t),y(t)], trange, inits,
window, stepsize=0.1, arrows=NONE);
```

1c. Based on the results in your plot, classify what type of critical point the origin is and whether the origin is stable or not.

1d. Use the `Eigenvectors` command to calculate the eigenvalues of the matrix of the system of part (a) and **discuss** the relation of this result to the graph in part (b) and the classification in part (c)

2a. Consider the almost linear system

$$x' = 2y - 2x + xy - x^2, \qquad y' = 4y + 4x - xy - x^2.$$

The equilibrium solutions are $[x = 0, y = 0]$, $[x = -2, y = 2]$, and $[x = 4, y = 4]$. Maple can obtain these by using

```
F2:=2*y-2*x+x*y-x^2; G2:=4*y+4*x-x*y-x^2;
eqpts:=solve({F2,G2},{x,y});
```

Use the `DEplot` command to obtain a plot of the direction field of this system using the values of `trange` and `window` defined earlier.

2b. **Zoom in** on each equilibrium point. That is, for each equilibrium solution $(x_0, y_0)$, use `DEplot` to plot the direction field in the rectangle $x_0 - 1 \leq x \leq x_0 + 1$, $y_0 - 1 \leq y \leq y_0 + 1$. Add some typical trajectories to your plots which illustrate the behavior near each equilibrium solution (make sure the `stepsize` option is chosen properly). Using these plots, state whether each of the three equilibrium solutions is stable or unstable.

2c. The type and stability of the critical points can also be determined by examining the eigenvalues of the corresponding linear system. The entries of matrix of the linear system corresponding to each critical point can be obtained by Maple by executing the following sequence of commands, which calculate the partial derivatives of the right hand sides of the differential equations, assembles them into a matrix, and then substitutes the coordinates of the critical point for $x$ and $y$.

```
A11:= diff(F2,x);
A12:= diff(F2,y);
A21:= diff(G2,x);
A22:= diff(G2,y);
A:= «A11 | A12>,<A21 | A22»;
Aa:= subs(eqpts[1],A);
Eigenvectors(Aa);
```

You should find the other **linearizations** in a similar fashion using the names `Ab` and `Ac` for the matrices.

2d. **Discuss** the relation between the graphs found in part (b) and the algebraic results in part (c).

End of Lab 5