Math 244, Practice problems for Exam 2

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Problem 1: 2nd order Solve the following differential equations:

1. $y'' - 2y' + y = 0$.
2. $y'' - 2y' - 3y = 3e^{2t}$, $y(0) = 1$, $y'(0) = 1$.
3. $y'' - 2y' - 8y = 17\cos(t)$.
4. $y'' + 3y' + 2y = 16t^3 + 4t^2$.
5. $y'' + y' + 8y = e^x \cos(2x)$.
6. $y'' - 2y' + y = \frac{2e^t}{1+t^2}$.
7. $y'' + 9y = \cos^2(3t)$.
8. $16y'' - 8y' + 145y = 0$, $y(0) = -2$, $y'(0) = 1$.
9. $t^2y'' - ty' = t$, $y(1) = 1$, $y'(1) = 0$.

Problem 2: Reduction of order Consider the equation $t^2y'' + 3ty' + y = 0$, $t > 0$. Verify that $y_1(t) = 1/t$ is a solution. Then, use the method of reduction of order to find a second solution $y_2(t)$ of the given equation.

Problem 3: Abel's theorem Consider the functions $y_1(t) = e^t$ and $y_2(t) = t$. Compute the Wronskian of these functions. Are they linearly independent? Without solving the equation, explain why they cannot be solutions of the following differential equation: $y'' - t^2y + 3ty = 0$.

Problem 4: Oscillations without damping A mass of 100 grams stretches a spring 5cm. If the mass is set in motion from its equilibrium position with a downward velocity of 10cm/sec, and if there is no damping, determine the deviation u of the position from the equilibrium position at any time t. When does the mass first return to its equilibrium position?

Problem 5: Oscillations with damping A spring is stretched 1/6m by a force of 1N. A mass of 3kg is hung from the spring and is also attached to a viscous damper that exerts a force proportional to the velocity. If the velocity of the mass is 2m/sec then the damper exerts a force of 12 newtons. Write the differential equation corresponding to the system and calculate the values of the coefficients. If the mass is pulled down 5cm below its equilibrium and given an initial velocity of 10m/sec, determine its position u (displacement from the equilibrium position) as a function of time t.

Problem 6: Equation $x'' + 4x' + kx = 0$ represents a mass-spring-dashpot system. For which values of the spring constant k will the solutions oscillate, and what will be their pseudoperiod?

Problem 7: Systems Consider the following linear first-order system:

$$x'(t) = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} x(t)$$

Find the eigenvalues and eigenvectors of the given system. Find the general solution. If $x_1(0) = 1$, $x_2(0) = 3$, solve the initial value problem. Draw the phase portrait and explain the stability of this system.

Problem 8: Systems Consider the following linear first-order system:

$$bfx'(t) = \begin{pmatrix} 3 & 0 & -2 \\ 4 & 1 & -4 \\ 4 & 0 & -3 \end{pmatrix} bfx(t)$$

Find the eigenvalues and eigenvectors of the given system. Find the general solution. If $x_1(0) = 2$, $x_2(0) = 5$, $x_3(0) = 2$, solve the initial value problem. How does the solution behave as t goes to infinity?

Problem 9: Systems–complex eigenvalues Consider the following linear first-order system:

$$x'(t) = \begin{pmatrix} -3 & 1 \\ -4 & -3 \end{pmatrix} x(t)$$

Find the eigenvalues and eigenvectors of the given system. Find the general solution. If $x_1(0) = 2$, $x_2(0) = 2$, solve the initial value problem. Draw the phase portrait and explain the stability of this system.

Problem 10: For each of the following differential equations, consider the behaviour of the general solutions:

1. $y'' + 2y' + 5y = e^{-3x}$.
2. $y'' + y = \sin x$.
3. $x^2y'' - 4xy' + 4y = 0$.
4. $x^2y'' - 4xy' + 4y = \frac{1}{x^2}$.

Write next to each equation the letter which best describes the solutions: A. All solutions tend to 0 as x goes to $+\infty$; B. At least one solution tends to 0 as x goes to $+\infty$, but not all solutions have this property; C. none of the above.

Problem 11: Higher order Find the general solution to $y^{(5)} - 2y^{(3)} + 2y'' - 3y' + 2y = 0$.

Problem 12: Higher order Indicate the form in which you would be looking for a solution to

$$y^{(4)} + 2y^{(3)} + 2y'' + y' = 3e^t + 2te^{-t} + e^{-t}\sin(t).$$

Do not solve the equation!

Problem 12: Find the general solution to $y^{(6)} + y^{(3)} = t$.

Problem 13: More on geometry of systems The system

$$x'(t) = \begin{pmatrix} \alpha & 2 \\ -2 & 0 \end{pmatrix} x(t)$$

contains a parameter α. Describe how the solutions depend qualitatively on α; in particular, find the critical values of α at which the qualitative behaviour of the trajectories in the phase plane changes markedly.

Problem 14:Euler Knowing that $x/2 + x^2$ and $x/2 - 3x^3$ are two particular solutions of

$$x^2y'' + P(x)y' + 6y = x,$$

find $P(x)$. What is the homogenous solution to this equation?

Problem 15: Application Two identical cubical cells, containing salt solution and separated by a semi-permeable membrane, are surrounded by a continuously flowing water bath containing no salt, and separated from it by another semi-permeable membrane which surrounds each cell. Let $x_1(t)$ and $x_2(t)$ be the respective amounts of salt in the two cells at time t.

1. Assume that the salt flow across a membrane is proportional to the concentration difference on either side of the membrane. Take 1 to be the constant of proportionality for the membrane between the cells, and 2 to be the contant for the external membrane. Show that x_1 and x_2 satisfy the system

$$x'(t) = \begin{pmatrix} -3k & k \\ k & -3k \end{pmatrix} x(t)$$

   where k is a constant related to the volume of the cells.
2. Take $k = 1$. Find the general solution to the system above.
3. Find the solution satisfying the initial condition $x_1(0) = 2$, $x_2(0) = 0$. For this initial condition, at what time will the x_1 cell contain twice the salt of the x_2 cell?

Problem 16: Application Review the tank problems from the lecture notes and homework!