Mathematics 244 Maple Lab 2

Numerical Methods

Spring 2006

Section 0: Introduction

Note: this document has few changes since Fall 2005, and that version contained some **cosmetic** changes from Fall 2004 version along with **clarifying statements** about the goals of the project.

In this lab, we shall use Maple's ability to approximate the solution of differential equations by numerical methods and add these solutions to the direction fields that we studied in Lab 1. This allows us to extend our understanding of the solutions of differential equations to equations for which a closed form solution is not available.

Please turn in only the printout of your main Maple worksheet. Include explicit answers to all questions asked, using the **text** feature of Maple to insert them in the worksheet. The final worksheet should be **edited** to remove any extraneous material. Editing may be guided by the **Print Preview** feature available from the File menu or toolbar. In particular, this will identify places where you can use the **Insert** menu to add a **Page Break** to avoid an unsuitable automatic break.

The first step is to obtain the **seed file** from the web page and arrange to save it in your directory on eden. You will be asked to execute the Maple commands that have been placed in this worksheet and to add some of your own. The ease of obtaining the seed file means that verbatim quotes of Maple instructions in the descriptions can be phased out, so not all prepared instructions will be described here.

There is also a **supplementary worksheet** that suggests additional experiments that may lead to better answers to the questions considered in this project.

The file begins with the line `with(plots):  with(DEtools):` in a preliminary section as in previous projects.

Equation 1

The sequence of commands to begin working with Equation 1 have already been entered into the seed file. Execute them to see how Maple can be used to plot the direction field along with the particular solution satisfying this differential equation and the initial condition $y(0) = 0$. The quantities `x1int` and `y1int` are introduced to have standard names for the **ranges** of the variables in **all** graphs in this problem. For the other exercise, corresponding commands can be constructed by copying lines to different places in a worksheet and doing some minor editing.

```
de1:= diff(y(x),x) = 3 + 5*x*y(x)
initval1a:={[y(0)=0]};
x1int:=x=-1..1;y1int:=y=-3..3;
DEplot(de1, y(x), x1int, initval1a,y1int,title="Default view of equation 1");
```

The `DEplot` command has many options which you can read about in the Maple help pages. In particular, it is possible to specify the color of the solution curve by using `linecolor`, the color of the direction field by using `color`, to make the gridwork of lines in the direction field finer (or coarser) by using the `dirgrid` option, and to get a finer resolution in computing the solution with the `stepsize` option. For example, try:

```
opt1:=linecolor = black,color = green, dirgrid= [30,30],stepsize=0.1;
DEplot(de1, y(x), x1int, initval1a,y1int, opt1);
```

By varying the initial conditions, it is possible to approximate special solutions. For example, one type of solution of this equation, e.g. the one with with $y(0) = -1$, is everywhere increasing, whereas another type, e.g. the one with $y(0) = -2$ reaches a maximum value and then becomes rapidly decreasing as $x \to +\infty$. By plotting solutions with other values of $y(0)$ between -2 and -1, you can identify more precisely where this change of behavior occurs. The **supplementary worksheet** contains instructions to graph the solutions with a selection of values of $y(0)$. In the first, $y(0)$ takes **all** the values -2.0, -1.9, -1.8, -1.7, -1.6, -1.5, -1.4, -1.3, -1.2, -1.1, -1.0. This includes graphs of both types and illustrates the role of the initial condition. While this is useful for **studying the property**, it gives **too much detail for a report**. The second uses only the values -2.0, -1.5, -1.0, and used a new `linecolor` option to override the setting in `opt1` and assign different colors to the different solutions.

Use this graph to select the **two** initial values giving solutions **exactly one solution** of each type. Modify the instruction to show **only** these two solutions and copy the **modified instruction** to the main worksheet and execute it there to get a graph with one solution of each type. You may also plot the curves in different colors to aid in identifying them.

Many qualitative properties of solutions of a differential equation can be determined from the equation itself. These qualitative properties serve as a check against obvious errors in numerical solutions. While Maple selects a fairly reliable procedure for solving initial value problems numerically, all numerical work must balance accuracy against computational effort, and sometimes a proper balance is not achieved. Simple methods to detect computed results that violate fundamental properties of solutions are the first defense against misplaced trust of computed results. For the equations in these projects, you should be able to achieve **visually accurate** results with **no noticeable computational effort**.

For this equation, the two types of solutions that we found can be partially detected by features of the equation. Conclude your study of this equation with a **discussion** of the following features of the equation related to the following properties of solutions:

1. In the quadrant where $x > 0$ and $y > 0$, solutions seem to be **steadily increasing**. How can this be proved from the differential equation?
2. Other solutions seem to have a **unique maximum point**. Such points must be **local extrema** which are characterized in Calculus. How can the differential equation use this characterization to identify local extrema? The equation can also be used to get information about second derivatives that will show that the local extrema in the fourth quadrant are all local maxima. The supplementary worksheet shows one way to get this information, but you need not include any discussion of this point in your main worksheet.
3. Refine this use of the equation to identify a curve that contains all maximum points of these curves. To check your answer, a `display` could be constructed in the supplementary worksheet that adds this curve to the previous graph of several solutions to the equation. Note that this curve is **not** a solution; its role is only to identify a useful feature of the solutions that should be shown accurately in the graph of a solution. This role can best be illustrated by drawing it in a different color than used for solutions or the direction field. Although this plot is not to be submitted as part of this project, such curves will appear later in the course.

Equation 2

Consider the differential equation

$$\frac{dy}{dx} = \left(3x^2 - y^2\right)\sin y$$

with the initial condition $y(0) = -1$.

Introduce the name `de2` for this equation and use the `DEplot` command to plot a direction field and the (numerical) solution of this initial value problem for $-3 \le t \le 6$ and $-4 \le y \le 0$ with **no special options**. Although the result **will not be satisfactory**, you should **leave this graph in your worksheet** since you will discuss its features..

To get a more accurate graph, experiment with different stepsizes. The **supplementary worksheet** contains the choices `opt2:=linecolor=black, color=green, dirgrid=[24,12], stepsize=0.1` and `opt3` with `stepsize=0.05` and the same values of the other options. Copy the commands leading to the `DEplot` command that you just constructed from the main worksheet to the supplementary worksheet, and add one of these options. Try other choices of `stepsize` and select one that gives **what you think is an accurate graph** while not using an excessively small `stepsize`. The plot should **look smooth**, and no there should be **no delay** in computing the solution for the plot. You may also experiment with changing other options. When you have selected a suitable graph, copy the instructions that you need to produce the graph to the main worksheet. The main worksheet will now contain exactly two graphs based on this equation. The second plot should use a small enough `stepsize` to give a smooth graph that avoids the flaws of the first graph, but not so small that you notice the time needed to compute the solution.

You are now ready to **discuss** what you have learned about this equation.

(1) **Verify** that the constant functions $y = 0$ and $y = -\pi$ are solutions of the equation. Note that you don't need to know anything about how to solve an equation to recognize a solution if it is given to you. You can do this with Maple if you wish, but other methods may be easier. However you do this, include a **text summary** of how you checked this.

(2) How does this show that the solution of the given initial value problem has $0 > y(t) > -\pi$ for all t? (Answer in **text**.)

(3) How do you know that the first plot is **not** correct? What gives you confidence that your second plot accurately shows the shape of the graph of the solution of this equation?

End of Lab 2