Remember that no books, notes, or calculators may used during an actual exam. You will be given these formulas:

$$g = 9.8\ \text{m/sec}^2 \qquad u_1' = \frac{-y_2 g}{W[y_1, y_2]} \qquad u_2' = \frac{y_1 g}{W[y_1, y_2]} \qquad v'' + \big(p + 2\frac{y_1'}{y_1}\big)v' = 0.$$

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1. Find the general solution of $y'' - 2y' + 2y = x^2 + 2 + 2\cos x$ *by the method of undetermined coefficients.* Express your answer in terms of real quantities.

2. Find the general solution of the equation $x^2y'' + 2xy' - 6y = x^2$.

3. Solve the initial value problem for $\mathbf{x}(t)$: $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$, where $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 5 & 6 \\ 0 & -3 & -4 \end{pmatrix}$.

4. Let $A = \begin{pmatrix} 1 & -3 \\ 2 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -1 & 0 \\ 1 & -3 & 4 \end{pmatrix}$, and $\mathbf{b} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ Compute AB and solve $A\mathbf{x} = \mathbf{b}$.

5. A mass of 1 kg hangs from a spring of constant 8 newton/meter.

(a) How much is the spring stretched when the mass is in its equilibrium position?

(b) The mass is released from 1 meter above its equilibrium position with a *downward* speed of 6 meter/sec. Find the subsequent motion and determine (i) the first time that mass reaches its equilibrium position, and (ii) how far down the mass goes. Ignore friction.

(c) The system is now placed in heavy oil, introducing a damping constant of 6 newton-sec/meter, and the mass is released in the same way. Determine whether or not the equilibrium position is reached in finite time.

(d) The mass in the system of part (c) is now subject to a periodic external force of value $3\cos t$ newton. Find the amplitude of the oscillations of the mass after a long time.

6. Solve $y'' - 6y' + 9y = 0$; $\quad y(0) = 1$, $\quad y'(0) = -2$.

7. Determine whether or not the three vectors $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} -1 \\ 6 \\ 1 \end{pmatrix}$, $\mathbf{x}_3 = \begin{pmatrix} -1 \\ 10 \\ 5 \end{pmatrix}$ are linearly independent. Justify your answer.

8. Find the general solution of the equations $x' = x + y$; $\quad y' = 4x - 2y$, and give a careful drawing of the phase plane (xy-plane) for this system, showing various typical and special trajectories.

9. Redo exercises 2, 3, and 4 from the *Notes on Linear Algebra* on the web page.

10. Suppose that A is a 20×20 matrix, that $\mathbf{b}$ is a column vector with 20 components, and that the equations $A\mathbf{x} = \mathbf{b}$ have no solution. Explain why the equations $A\mathbf{x} = \mathbf{0}$ **must** have a nontrivial solution.

11. Let $\mathbf{x}' = P\mathbf{x}$ be a system of equations for the unknown function $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, where $P = P(t)$ is a 2×2 matrix whose coefficients are functions of t which are defined and continuous for all t. Let $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ be two solutions of this system. (a) Show that $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$ is also a solution, for c_1 and c_2 constants. (b) Define the Wronskian $W(t)$ of $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$. (c) Show that if $W(0) \neq 0$ then a solution of the form given in (a) can be used to solve any initial value problem $\mathbf{x}' = P\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}^0$.

ANSWERS (not checked too carefully—be wary):

1. $y(x) = Ce^x \cos x + De^x \sin x + x^2/2 + x + 3/2 + (2/5)\cos x - (4/5)\sin x$.

2. $y(x) = c_1x^{-3} + c_2x^2 + (1/5)x^2 \ln x$.

3. $\mathbf{x}(t) = -11e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 4e^{-t} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - 3e^{2t} \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$. $\quad$ 4. $\begin{pmatrix} -1 & 8 & -12 \\ 8 & -14 & 16 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} -1/5 \\ -2/5 \end{pmatrix}$.

5. (a) 1.225 meter; (b) $u(t) = -\cos 2\sqrt{2}t + (6/2\sqrt{2})\sin 2\sqrt{2}t$ meter; $t_0 = (1/2\sqrt{2})\arctan(\sqrt{2}/3)$ sec.
(c) Yes, at time $t = \ln(\sqrt{2})$ sec ; (d) $3/\sqrt{85}$ meter.

6. $e^{3t} - 5te^{3t}$. $\quad$ 7. No. $\quad$ 8. $c_1e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2e^{-3t}\begin{pmatrix} 1 \\ -4 \end{pmatrix}$; saddle point.

9, 10. See Linear Algebra notes for some answers. $\quad$ 11. See book or class notes.