

The exam will be held **Wednesday, May 9, 12:00-3:00 PM, in SEC 117**. No books, notes, or calculators may be used during the exam.

**Coverage notes:** *Section 2.6:* Exact equations YES, integrating factors NO.

*Section 3.5:* Double root for the characteristic equation YES, reduction of order in general NO.

*Section 7.7:* NO.

*Everything else in the course:* YES.

You will be given the following formula sheet.

#### Formulas

Integrating factor:  $\mu = e^{\int p(x) dx}$ .

Variation of parameters:  $u_1' = \frac{-y_2 g}{W(y_1, y_2)}$ ,  $u_2' = \frac{y_1 g}{W(y_1, y_2)}$ .

Complex numbers:  $e^{i\theta} = \cos \theta + i \sin \theta$ .

Infinite series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\frac{1}{(1-x)} = 1 + x + x^2 + x^3 + \dots$$

This review sheet contains primarily problems on material covered since the second exam, although there are a few problems on earlier material. However, the exam will be cumulative. **For a complete review you should use also the problem sheets for exams 1 and 2**, still available on the web page; reviewing exams 1 and 2 themselves may also be useful.

1. Consider the equations  $x' = x(x + y - 4)$ ,  $y' = y(1 + 2x - y)$ .

(a) Show that  $x = 1$ ,  $y = 3$  is a critical point, find the linearized equations near this point, and thus determine whether or not this critical point is stable, asymptotically stable, or unstable. What is the *type* of this critical point?

(b) Find all other critical points.

(c) Sketch the first quadrant  $x \geq 0$ ,  $y \geq 0$  of the phase plane, indicating, by arrows or otherwise, regions where  $x$  and  $y$  are increasing,  $x$  is increasing and  $y$  decreasing, etc., and where the trajectories are horizontal and vertical.

(d) For each initial condition below, determine (from your sketch)  $\lim_{t \rightarrow \infty} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ :

(i)  $x(0) = 0$ ,  $y(0) = 3$ ; (ii)  $x(0) = .5$ ,  $y(0) = 2$ .

2. Consider the system  $\mathbf{x}' = A\mathbf{x}$ , where  $A = \begin{pmatrix} 0 & -1 \\ 5 & 2 \end{pmatrix}$ . (a) What is the type and stability of the critical point at the origin? (b) Find a solution (in terms of real functions) which satisfies  $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ .

3. (a) Find the general solution of the equations  $x' = 3x - y$ ;  $y' = 4x - 2y$ . (b) Give a careful drawing of the phase plane ( $xy$ -plane) for this system, showing various typical and special trajectories.

(c) Find initial conditions  $(x_0, y_0)$  such that if a solution satisfies  $x(0) = x_0$ ,  $y(0) = y_0$  then

(i)  $\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} y(t) = 0$ ; (ii)  $\lim_{t \rightarrow -\infty} x(t) = \lim_{t \rightarrow -\infty} y(t) = 0$ ;

(iii)  $\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow -\infty} x(t) = -\infty$ .

4. Redo problem 3, parts (a) and (b), for the equations  $x' = x + y$ ;  $y' = -4x + 6y$ .

5. (a) Solve  $y'' + 2xy' + 4y = 0$  as a series  $y = \sum_{n=0}^{\infty} a_n x^n$ , finding all coefficients up to order  $x^7$  and displaying your final answer in the form  $y(x) = a_0 y_0(x) + a_1 y_1(x)$ .  
 (b) Find the solution of the equation which satisfies  $y(0) = 3, y'(0) = 1$ .  
 (c) Find the general formula for  $a_n$  and then find either  $y_0$  or  $y_1$  in closed form—i.e., sum the corresponding series.
6. The equation  $P(x)y'' + x^2y' + (x^5 - 1)y = 0$  is to be solved in a power series of the form  $y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ . For each case below, determine whether  $x_0$  is an ordinary point (so that such a solution is possible) and, if it is, find the minimum possible radius of convergence of the resulting series.  
 (i)  $P(x) = \cos x, x_0 = 0$ ; (ii)  $P(x) = x^2 - 2x + 5, x_0 = 3$ ;  
 (iii)  $P(x) = e^x, x_0 = 10$ ; (iv)  $P(x) = \sin x, x_0 = 0$ .
7. In (a) and (b) below give a matrix  $R$  in row-echelon form satisfying the given condition, or explain why this is impossible.  
 (a)  $R$  is a  $4 \times 3$  matrix. The equations  $R\mathbf{x} = \mathbf{0}$  have a unique solution.  
 (b)  $R$  is a  $4 \times 4$  matrix. For every  $\mathbf{b}$  the equations  $R\mathbf{x} = \mathbf{b}$  have a solution containing a free parameter.
8. A certain time-dependent physical variable  $y$  is believed to satisfy the differential equation  $dy/dt = \sin y$ .  
 (a) Find all equilibrium values of  $y$  and classify each as stable or unstable. (b) What is the eventual value of  $y$  (i.e.,  $\lim_{t \rightarrow \infty} y(t)$ ) if initially (i)  $y = 2$ , (ii)  $y = 0$ , (iii)  $y = -0.001$ ? (c) A alternate model predicts that  $y$  should satisfy the equation  $dy/dt = \sin y + a$ , for some positive constant  $a$ . For what values of  $a$  will this second model behave in a way which is *drastically* different from the first model?
9. Let  $\mathbf{x}' = P\mathbf{x}$  be a system of two equations in two unknowns, where  $P$  is a  $2 \times 2$  matrix whose coefficients are continuous functions defined for all  $t$ . Let  $\mathbf{x}^{(1)}(t)$  and  $\mathbf{x}^{(2)}(t)$  be two solutions of this system.  
 (a) Show that  $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$  is also a solution, for  $c_1$  and  $c_2$  constants.  
 (b) Define the Wronskian  $W(t)$  of  $\mathbf{x}^{(1)}$  and  $\mathbf{x}^{(2)}$ , and show that if  $W(0) \neq 0$  then (a) can be used to solve any initial value problem  $\mathbf{x}' = P\mathbf{x}, \mathbf{x}(0) = \mathbf{x}^0$ .  
 (c) Give an example in which  $W(0) = 1$  and  $W(2) = 0$ , or explain why this is impossible.
10. (From Exam 2): Determine if each statement is true or false, or if the given information is insufficient to tell.  
 (a) If  $n > m$  then the equations have a solution for any  $\mathbf{b}$ .  
 (b) If  $n > m$  then the homogeneous equations  $A\mathbf{x} = \mathbf{0}$  have a nonzero solution.  
 (c) If  $n < m$  then for some  $\mathbf{b}$  the equations have no solution.  
 (d) If  $n = m$  and  $A\mathbf{x} = \mathbf{b}$  has a solution for every  $\mathbf{b}$ , then  $\det A = 0$ .  
 (e) If  $n = m$  and  $A\mathbf{x} = \mathbf{b}$  has a solution for every  $\mathbf{b}$ , then  $A\mathbf{x} = \mathbf{0}$  has only the zero solution.  
 (f) If  $n = m$  and  $A\mathbf{x} = \mathbf{0}$  has a nonzero solution then  $\text{rank } A = n$

ANSWERS (not checked too carefully—be wary):

1. (a)  $\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$ ; unstable (saddle point). (b)  $(0, 0), (0, 1), (4, 0)$ . (c) Both  $(0, 1)$ .
2. (a) Spiral source (unstable spiral); (b)  $y(t) = (e^t/2) \begin{pmatrix} 2 \cos 2t - 3 \sin 2t \\ 4 \cos 2t + 7 \sin 2t \end{pmatrix}$ .
3. (a)  $c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ . (c) (i)  $(1, 4)$ ; (ii)  $(1, 1)$ ; (iii)  $(-1, -2)$ .
4. (a)  $c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ .
5. (a)  $y = a_0 y_0 + a_1 y_1, y_0 = 1 - 2x^2 + 4x^4/3 - 8x^6/15 + \dots, y_1 = x - x^2 + x^5/2 - x^7/6 + \dots$   
 (b)  $a_0 = 3, a_1 = 1$ . (c)  $a_{2k} = (-2)^k / [(2k - 1) \cdot (2k - 3) \cdots 3 \cdot 1], a_{2k+1} = (-1)^k / k!, y_1 = x e^{-x^2}$ .
6. (i)  $\rho = \pi/2$ . (ii)  $\rho = 2\sqrt{2}$ . (iii)  $\rho = \infty$ . (iv) Not ordinary point.
7. (a) Possible. (b) Impossible.
8. (a) Unstable:  $0, \pm 2\pi, \pm 4\pi, \dots$ ; Stable:  $\pm\pi, \pm 3\pi, \pm 5\pi, \dots$  (b)  $\pi, 0, -\pi$ . (c)  $a > 1$ .
9. See book or class notes.
10. (a) ?, (b) T, (c) T, (d) F, (e) T, (f) F.