The Phase Plane for some Linear Systems

The figures on the following pages show the phase plane for certain linear, homogeneous, constant coefficient systems of ODE's in two unknowns:

$$\mathbf{x}' = A\mathbf{x}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

The direction field of the differential equation is shown by small green arrows. Typical trajectories of the system are shown in black; for the saddle point and stable node, the special trajectories along straight lines (parallel to the eigenvectors) are shown in red or blue. The direction of flow along each trajectory can be inferred from the arrows in the direction field at nearby points.

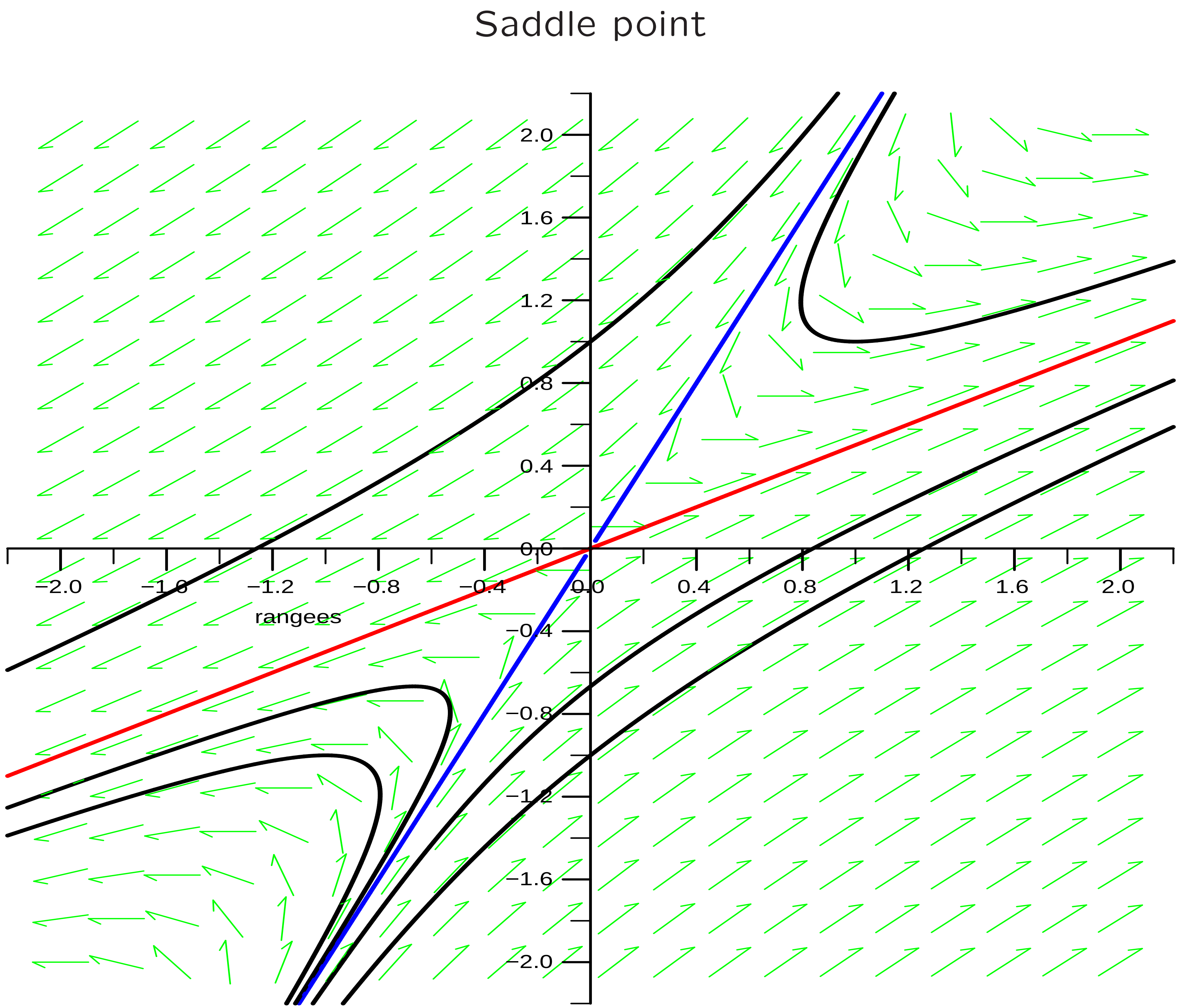
The specific equations whose solutions are plotted:

Saddle point: $A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}, \quad \mathbf{x}(t) = c_1 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Stable node: $A = \begin{pmatrix} -5 & 2 \\ -4 & 1 \end{pmatrix}, \quad \mathbf{x}(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

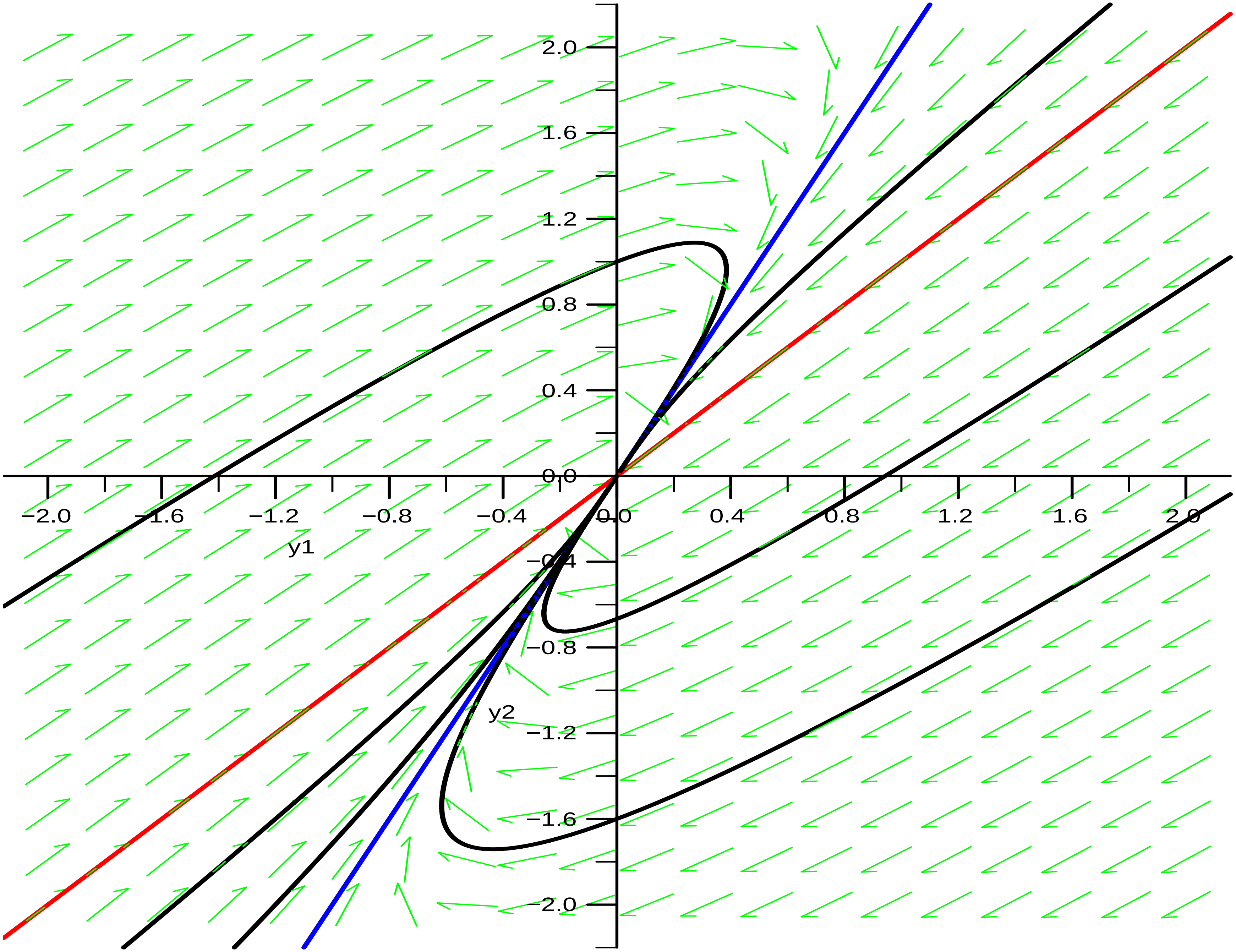
Unstable spiral: $A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix},$

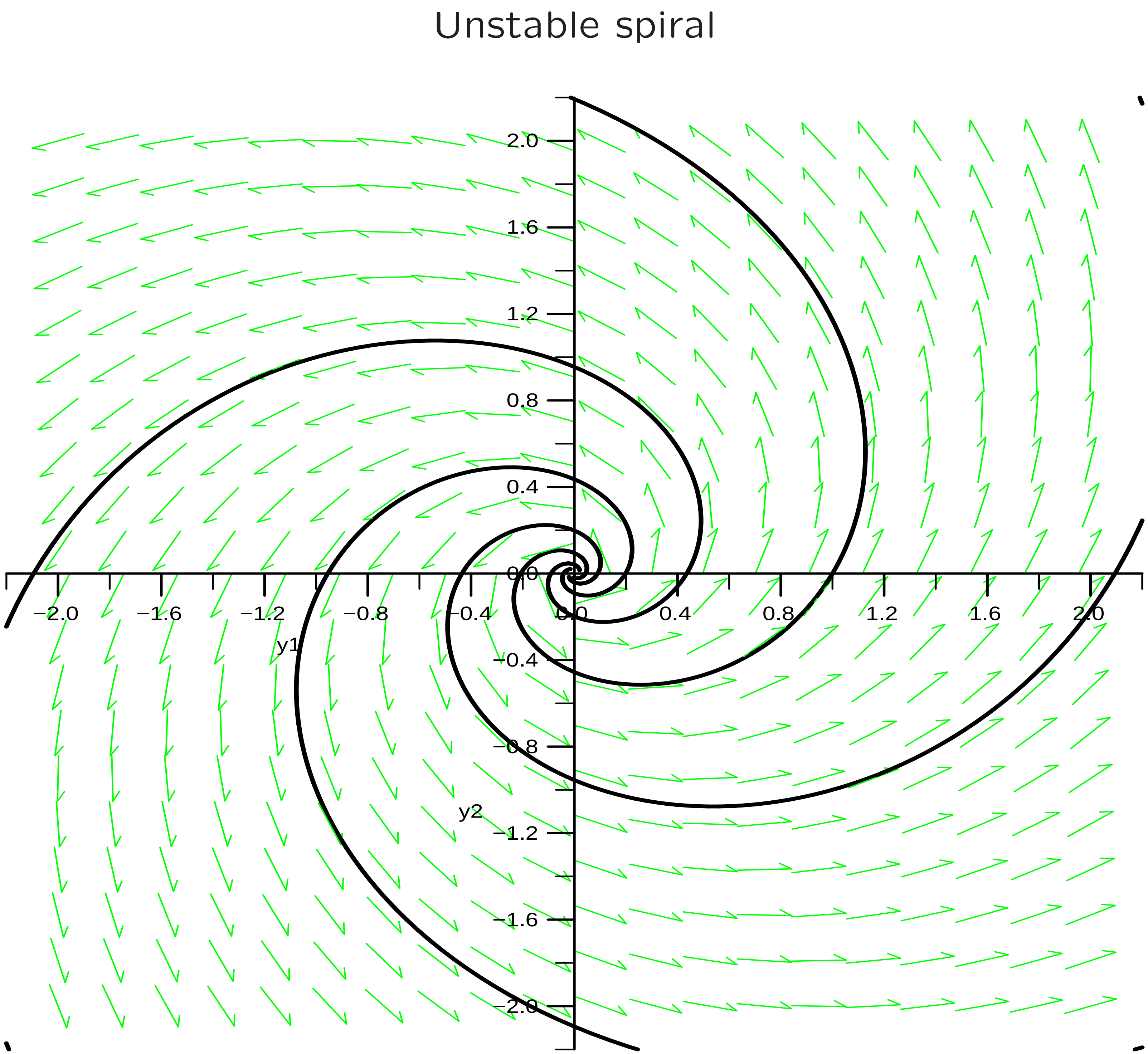
$$\mathbf{x}(t) = c_1 e^{t} \begin{pmatrix} -\sin 2t \\ \cos 2t \end{pmatrix} + c_2 e^{t} \begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix}$$



Saddle point
2.0
1.6
1.2
0.8
0.4
0.0
−0.4
−0.8
−1.2
−1.6
−2.0
−2.0
−1.6
−1.2
−0.8
−0.4
0.0
0.4
0.8
1.2
1.6
2.0
rangees

Stable node




Unstable spiral
2.0
1.6
1.2
0.8
0.4
0.0
−0.4
−0.8
−1.2
−1.6
−2.0
−2.0
−1.6
−1.2
−0.8
−0.4
0.0
0.4
0.8
1.2
1.6
2.0
y1
y2