

LAB 5: Orthogonal Subspaces, QR Factorization, and Inconsistent Linear Systems

In this lab you will use MATLAB to study four topics:

- The *Gram-Schmidt Algorithm* that transforms a given set of basis vectors into an orthonormal basis, and the associated $A = QR$ matrix factorization.
- The orthogonal projection of a vector onto a subspace
- The best approximate solution to an inconsistent system $A\mathbf{x} = \mathbf{v}$.
- The orthogonal decomposition of \mathbf{R}^m associated with an $m \times n$ matrix A .

Reading from Textbook: In connection with this Lab, read through Sections 6.8, 6.9, 7.1, 7.2, 8.1, and 8.2 of the text and work the suggested problems for each section.

Tcodes and Script Files: For this lab you will need the Teaching Codes

`grams.m, linefit.m, lsq.m, partic.m`

Before beginning work on the Lab questions you should copy these codes from the Teaching Codes directory on the Math Department/Course Materials/Linear Algebra 250 web page to your diskette or hard drive (see Lab 3 for more details).

You will also need the m-files `rmat.m` and `rvect.m` from Lab 3. These files should already be on your diskette or in your hard-drive. If you didn't do Lab 3, get a copy of that lab assignment and create these files as indicated there.

Lab Write-up: You should open a diary file at the beginning of each MATLAB session (see Lab 1 for details). Be sure to answer all the questions in the lab assignment.

Random Seed: When you start your MATLAB session, initialize the random number generator by typing

`rand('seed', abcd)`

where $abcd$ are the last four digits of your Student ID number. This will ensure that you generate your own particular random vectors and matrices.

BE SURE TO INCLUDE THIS LINE IN YOUR LAB WRITE-UP

The lab report that you hand in must be your own work. The following problems all use randomly generated matrices and vectors, so the matrices and vectors in your lab report will not be the same as those of other students doing the lab. Sharing of lab report files is not allowed in this course.

Question 1. Gram-Schmidt Orthogonalization and QR Factorization

(a) **Dot Products:** Generate three random vectors in \mathbf{R}^3 by

$$u_1 = \text{rvect}(3), u_2 = \text{rvect}(3), u_3 = \text{rvect}(3)$$

Calculate the dot products of these vectors:

$$u_1' * u_2, u_1' * u_3, u_2' * u_3$$

Are the vectors mutually orthogonal?

(b) **Gram-Schmidt Orthogonalization:** Let u_1, u_2, u_3 be the three vectors from part (a). You will now change it into an orthonormal set, following the algorithm in the proof of Theorem 6.18 on page 308 of the text. Set

$$v_1 = u_1, v_2 = u_2 - ((u_2' * v_1) / (v_1' * v_1)) * v_1$$

Calculate the dot product of the vectors v_1 and v_2 . Are these vectors mutually orthogonal (within a negligible numerical error)? Now set

$$v_3 = u_3 - ((u_3' * v_1) / (v_1' * v_1)) * v_1 - ((u_3' * v_2) / (v_2' * v_2)) * v_2$$

(use the up-arrow key to edit the previous command). Calculate the dot products $v_1' * v_3$ and $v_2' * v_3$. Are the vectors v_1, v_2, v_3 mutually orthogonal (within a negligible numerical error)? See Figure 6.11 on page 309. Now rescale the vectors v_1, v_2, v_3 to have length one by using the MATLAB function `norm`:

$$w_1 = v_1 / \text{norm}(v_1), w_2 = v_2 / \text{norm}(v_2), w_3 = v_3 / \text{norm}(v_3)$$

Check (by MATLAB) that $\|w_i\| = 1$ for $i = 1, 2, 3$. Explain why the vectors w_1, w_2, w_3 are mutually orthogonal.

(c) **A = QR Factorization:** Set

$$Q = [w_1, w_2, w_3]$$

Use MATLAB to calculate $Q' * Q$. Explain why you could have predicted the answer from the result of part (b). What is the *inverse matrix* Q^{-1} ? Answer first without MATLAB and then check with MATLAB.

Set

$$A = [u_1, u_2, u_3], R = Q' * A$$

Explain the pattern of zeros in R from the orthogonality relations between the two sets of vectors $\{u_1, u_2, u_3\}$ and $\{w_1, w_2, w_3\}$. Use the formulas in part (b) to explain why the diagonal entries of R are positive. Verify by MATLAB that $A = Q * R$.

Question 2. Orthogonal Projections

(a) **Projection onto a Subspace:** Generate three random vectors $v_1, v_2, v_3 \in \mathbf{R}^5$ and form the matrix A with these vectors as columns:

$$v_1 = \text{rvect}(5); v_2 = \text{rvect}(5); v_3 = \text{rvect}(5); A = [v_1, v_2, v_3]$$

Let W be the subspace of \mathbf{R}^5 spanned by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Use the T-code `nulbasis.m` to calculate `nulbasis(A)`. From this calculation, determine whether the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent or linearly dependent, and explain your answer. What is $\dim(W)$?

Now use the T-code `grams.m` to obtain an orthonormal basis for W by the Gram-Schmidt process:

$$Q = \text{grams}(A); \mathbf{w}_1 = Q(:,1), \mathbf{w}_2 = Q(:,2), \mathbf{w}_3 = Q(:,3)$$

The orthogonal projection P from \mathbf{R}^5 onto the subspace W is then given by the 5×5 matrix

$$P = \mathbf{w}_1 \mathbf{w}_1' + \mathbf{w}_2 \mathbf{w}_2' + \mathbf{w}_3 \mathbf{w}_3'$$

(note that \mathbf{w}_1 is a 5×1 column vector; hence \mathbf{w}_1' is a 1×5 row vector and $\mathbf{w}_1 \mathbf{w}_1'$ is a 5×5 matrix). Explain by a hand calculation why

$$P\mathbf{v} = (\mathbf{w}_1 \cdot \mathbf{v})\mathbf{w}_1 + (\mathbf{w}_2 \cdot \mathbf{v})\mathbf{w}_2 + (\mathbf{w}_3 \cdot \mathbf{v})\mathbf{w}_3$$

for any vector $\mathbf{v} \in \mathbf{R}^5$ (see formula (1) on page 315 of the text). Verify that $P - P^T = 0$ and $P^2 - P = 0$ (these are the properties that characterize an orthogonal projection matrix).

Predict what the vectors $P\mathbf{v}_1, P\mathbf{v}_2, P\mathbf{v}_3$ should be, and then confirm your prediction by MATLAB.

(b) Orthogonal Decomposition $\mathbf{v} = \mathbf{w} + \mathbf{u}$: Generate another random vector $\mathbf{v} = \text{rvect}(5)$. Set $\mathbf{w} = P\mathbf{v}$, $\mathbf{u} = \mathbf{v} - \mathbf{w}$. Verify by MATLAB that $\mathbf{w}'\mathbf{u} = 0$. This shows that \mathbf{w} is in the subspace W and that \mathbf{u} is the component of \mathbf{v} perpendicular W (see Figure 6.13 on page 320 of the text).

Question 3. Approximate Solution to Inconsistent Linear System

(a) Best Approximate Solution to $A\mathbf{x} = \mathbf{v}$: Take the 5×3 matrix A as in part (a) of Question 2 and the vector $\mathbf{v} \in \mathbf{R}^5$ as in part (b) of Question 2. Explain how the decomposition $\mathbf{v} = \mathbf{w} + \mathbf{u}$ from part (b) shows that

(i) There is **no** vector $\mathbf{x} \in \mathbf{R}^3$ that satisfies the equation $A\mathbf{x} = \mathbf{v}$

(ii) There is a vector $\bar{\mathbf{x}} \in \mathbf{R}^3$ that satisfies the modified equation $A\bar{\mathbf{x}} = \mathbf{w}$.

Calculate the *Least-Squares Approximate Solution* $\bar{\mathbf{x}} = \mathbf{xls}$ in (ii) by

$$\mathbf{xls} = \text{inv}(A'*A)*A'*\mathbf{v}$$

(see Theorem 7.2 on page 331 of the text). Check by MATLAB that $A*\mathbf{xls} - \mathbf{w} = 0$.

(b) Least-Squares Property: Since $A\bar{\mathbf{x}} = \mathbf{w}$, it is the vector in W that is closest to \mathbf{v} . Thus the function $\|A\mathbf{y} - \mathbf{v}\|$ is *minimized* by choosing $\mathbf{y} = \bar{\mathbf{x}}$ (this is the *least-squares property* – see Theorem 6.23 on page 321 of the text). To illustrate this, generate a random vector $\mathbf{y} = \text{rvect}(3)$ in \mathbf{R}^3 and compare $\text{norm}(A*\mathbf{y} - \mathbf{v})$ with $\text{norm}(A*\mathbf{xls} - \mathbf{v})$. Repeat the calculation twice using \uparrow . Is the least-squares property satisfied in each case?

Question 4. Column Space and Null Space

(a) Column Space of A and Null Space of A^T : Let A be the 5×3 matrix from Question 3, and let W be the column space of A . Let $U = \text{Null}(A^T)$ be the null space of A^T . What is $\dim(W) + \dim(U)$? Use this to obtain $\dim(U)$.

(b) **Orthogonal Decomposition:** Use the MATLAB command `NT = null(A')` to obtain a matrix whose columns are an orthonormal basis for U . You already have an orthonormal basis for W (the columns of $Q = \text{grams}(A)$). Describe in words the relations between these two sets of basis vectors (see Theorem 6.22 on page 317). How can you use these vectors to obtain an orthonormal basis for \mathbf{R}^5 ? Check your answer using MATLAB and the 5×5 matrix $C = [Q, NT]$.

Optional Extra-Credit Question: Fitting a Line to Data Points

Read Section 7.2 of the text and work the suggested exercises for this Section.

(a) **Generating and Plotting Linear Data:** Define a column vector of ten equally-spaced t values

```
t = [1:10]'
```

Now generate a column vector of the corresponding values of the linear function $y = 4 + t$ and plot it by

```
y = 4 + t; linefit(t,y)
```

(Note the semicolon that suppresses the display of the y data). MATLAB should open a new window in which this line is plotted. Notice that the line passes exactly through each data point, and the 'best fit' equation is the given function $4 + t$. Print this graph by clicking on `file` in the figure tool bar, and then click on `print`. Include the printed copy in your lab write-up.

(b) **Linear Data with Random Noise:** Now add some random noise to each y data value in part (a):

```
y = 2 + 4*rand(10,1) + t; linefit(t,y)
```

These random data points don't lie on the line from part (a), even though they show the same general trend. The line that is plotted is the *best fitting* line; it minimizes the *mean square error* between the y coordinates on the line and the data values. How many data points are above the best-fitting line? How many data points are below the best-fitting line? The equation of the best line fitting the random data points is displayed above the graph. Print this graph (following the same procedure as in part (a)). Include the printed copy in your lab write-up.

Final Editing of Lab Write-up: After you have worked through all the parts of the lab assignment, you will need to edit your diary file. Remove all errors and other material that is not directly related to the questions. Your write-up should only contain the required MATLAB calculations and the answers to the questions. Preview the document before printing and remove unnecessary page breaks and blank space.