Revised 10/15/02

LAB 3: *LU* Decomposition and Determinants

In this lab you will use MATLAB to study the following topics:

- The *LU* decomposition of an invertible square matrix A.
- How to use the *LU* decomposition to solve the system of linear equations $A\mathbf{x} = \mathbf{b}$.
- The determinant of a square matrix, how it changes under row operations and matrix multiplication, and how it can be calculated efficiently by the LU decomposition.
- The geometric properties of special types of matrices (rotations, dilations, shears).

Preliminaries

Reading from Textbook: Before beginning the Lab, read through Sections 2.5, 3.1 and 3.2 of the text and work the suggested problems for these sections.

Tcodes: In this course you will use some instructional MATLAB *m-files* called *Tcodes.* To obtain any of these files, use a web browser and go to the Math Department Home page *http://www.math.rutgers.edu.* Click on *course materials*, then on *Math 250 Introduction to Linear Algebra* , and then on *MATLAB Teaching Codes.* You will see a directory of the m-files. Click on the particular m-files that you need. Then in the menu bar click on *Files* and *Save As.* Fill in the directory information that is requested.

For this lab you will need the Teaching Codes

```
cofactor.m,  splu.m  plot2d.m,  house.m
```

Before opening MATLAB to work on the Lab questions you should copy these codes to your directory by the method described above.

Lab Write-up: You should open a diary file at the beginning of each MATLAB session (see Lab 1 for details). Begin the diary file with the comment line

```
% Math 250 MATLAB Lab Assignment #3
```

Type `format compact` so that your diary file will not have unnecessary spaces. Put labels to mark the beginning of your work on each part of each question. For example,

```
% Question 1 (a) ...
        ⋮
% Question 1 (b) ...
```

and so on.

Be sure to answer all the questions in the lab assignment. Insert comments in your diary file as you work through the assignment.

Random Seed: When you start your MATLAB session, initialize the random number generator by typing

```
rand('seed', abcd)
```

where *abcd* are the last four digits of your Social Security number. This will ensure that you generate your own particular random vectors and matrices.

BE SURE TO INCLUDE THIS LINE IN YOUR LAB WRITE-UP

The lab report that you hand in must be your own work. The following problems all use randomly generated matrices and vectors, so the matrices and vectors in your lab report will not be the same as those of other students doing the lab. Sharing of lab report files is not allowed in this course.

Question 1. Row Operations and LU Factorization

In this problem you will use MATLAB to carry out elementary row operations and to obtain the matrix factorization $A = LU$ for a square 3×3 matrix A.

(a) Generate a random 3×3 matrix A using the m-file `rmat(m,n)` that you created in Lab 2:

```
A = rmat(3,3), U = A
```

Here U has the initial value A, but at the end of the LU algorithm U will be upper triangular.

(b) Use the MATLAB editor to create an m-file called `col1.m` with the following MATLAB commands:

```
L1 = eye(3);
L1(2,:) = L1(2,:) + (U(2,1)/U(1,1))*L1(1,:);
L1(3,:) = L1(3,:) + (U(3,1)/U(1,1))*L1(1,:);
L1
```

(notice the use of ; to suppress screen output of the intermediate results). Execute this file by typing `col1` at the MATLAB prompt. If you get a division by zero error message, go back to step (a) and generate another A and U. Describe in words how the matrix L_1 is obtained from row operations on the 3×3 identity matrix (remember that $U = A$ at the beginning of the algorithm). Use MATLAB to calculate L_1^{-1} (if M is a square matrix that has an inverse, then the MATLAB command `inv(M)` will calculate the inverse matrix M^{-1}). Describe in words how the matrix entries of L_1^{-1} are obtained from those of L_1.

Now use MATLAB to replace the current value of the matrix U by the new value $L_1^{-1} * U$ (remember that the command $X = Y$ in MATLAB means to replace the current value of the variable X by the current value of the variable Y). Describe in words how the new value of U is obtained from the old value of U by row operations.

(c) Use the MATLAB editor to create an m-file called `col2.m` with the commands

```
L2 = eye(3);
L2(3,:) = L2(3,:)+ (U(3,2)/U(2,2))*L2(2,:); L2
```

This will be used with the matrix U modified as in (b). Execute this file by typing `col2` at the MATLAB prompt. If you get a division by zero error message, go back to step (a) and start again (this is very unlikely to happen since A is completely random). Use MATLAB to calculate L_2^{-1}. Describe in words how the matrix entries of L_2^{-1} are obtained from those of L_2.

Now use MATLAB to replace the current value of the matrix U by the new value $L_2^{-1} * U$. Describe in words how the new value of U is obtained from the old value of U by row operations.

(d) Use MATLAB to get $L = L_1 * L_2$. Verify by MATLAB that $A = L * U$ (where U has the value from (c)). Describe in words how the entries of L are obtained from those of L_1 and L_2 (see the boxed statement on page 140 of the text).

Question 2. Using LU Factorization to Solve $A\mathbf{x} = \mathbf{b}$

(a) Inverting A, L and U: Use MATLAB to calculate the inverses of the matrices L and U that you obtained in Question #1. Which entries in `inv(L)` and `inv(U)` are *always* zero (no matter what random matrix A you generate)? Which entries in `inv(L)` are *always* 1? For the matrices L_1 and L_2, you found in Question #1 that the inverse matrices are simply obtained by putting a minus sign in front of the entries below the main diagonal. Does this method give the correct inverse matrix for L?

(b) Solving $A\mathbf{x} = \mathbf{b}$ using L and U (See Example 4 on page 140 of the text): Use the m-file `rvect.m` from Lab 2 to generate a random vector `b = rvect(3)`. Calculate the solution

```
c = inv(L)*b
```

to the triangular system $L\mathbf{c} = \mathbf{b}$. Then calculate the solution

```
x = inv(U)*c
```

to the triangular system $U\mathbf{x} = \mathbf{c}$. Finally, calculate $A\mathbf{x}$ and check that it is $\mathbf{b}$ (since the entries in $\mathbf{b}$ are integers, this should be obvious by inspection).

Question 3. The Determinant Function

(a) Cofactor Expansion: The Teaching Code m-file `cofactor.m` calculates the matrix of cofactors of a square matrix. Generate a random 4×4 integer matrix `a = rmat(4,4)`. Then use MATLAB to calculate the cofactor matrix `c = cofactor(a)`. Now use MATLAB to calculate the four sums

```
a(1,1)*c(1,1) + a(1,2)*c(1,2) + a(1,3)*c(1,3) + a(1,4)*c(1,4)
a(2,1)*c(2,1) + a(2,2)*c(2,2) + a(2,3)*c(2,3) + a(2,4)*c(2,4)
a(1,3)*c(1,3) + a(2,3)*c(2,3) + a(3,3)*c(3,3) + a(4,3)*c(4,3)
a(1,4)*c(1,4) + a(2,4)*c(2,4) + a(3,4)*c(3,4) + a(4,4)*c(4,4)
```

(use the uparrow key ↑ and edit the line to save retyping). Use Theorem 3.1 (page 178) and Theorem 3.4 (page 188) to explain the answers you get. Then check by using MATLAB to calculate $\det(a)$.

(b) Triangular Matrices: Generate a random 5×5 matrix A and a random upper triangular matrix U by

```
A = rmat(5,5), U = triu(A)
```

Calculate the product `A(1,1)*A(2,2)*A(3,3)*A(4,4)*A(5,5)` of the diagonal entries of A. Can you find $\det(A)$ from this single term? Explain, and then check by calculating $\det(A)$ with MATLAB.

Now calculate the product `U(1,1)*U(2,2)*U(3,3)*U(4,4)*U(5,5)`. Can you find $\det(U)$ from this single term? Explain, and then check by calculating $\det(U)$ with MATLAB.

(c) Row Operations: Generate a 5×5 random integer matrix `A = rmat(5,5)`. Then swap the first and second row of A to get the matrix B using the following commands:

```
B = A; B(2,:)  = A(1,:); B(1,:)  = A(2,:)
```

What is the relation between $\det(A)$ and $\det(B)$? Explain by general properties of the determinant function, and then check by calculating `det(A)` and `det(B)` by MATLAB.

Now let C be the matrix obtained from A by multiplying the first row of A by 10 and adding to the second row of A using the following commands:

```
C = A; C(2,:)  = A(2,:)  + 10*A(1,:)
```

What is the relation between $\det(A)$ and $\det(C)$? Explain by general properties of the determinant function, and then check by MATLAB.

Finally, let D be the matrix obtained from A by multiplying the first row of A by 10:

```
D = A; D(1,:)  = 10*A(1,:)
```

What is the relation between $\det(A)$, $\det(D)$, and $\det(10 * A)$? Explain using general properties of the determinant function, and then check by MATLAB.

(d) Multiplicative Property: Generate a random 5×5 integer matrix `A = rmat(5,5)`. Then set `A(1,1)=0; A(2,1) = 0`. The reduction of A to row echelon form can be expressed in terms of a matrix factorization as $PA = LU$. Here P is a *permutation matrix* that expresses the row interchanges that are needed to apply Gaussian elimination to A, and L and U give the LU decomposition of PA (read pages 144-146 of the text). For the modified matrix A, explain why P will not be the identity matrix. (Are any row interchanges needed to do the first step of Gaussian elimination on A?)

You can calculate the $PA = LU$ factorization by using the T-code `splu.m`:

```
[P, L, U, sign] = splu(A)
```

Here `sign` gives det(P), which is +1 for an even number of row interchanges to transform P into the identity matrix, and -1 for an odd number of row interchanges.

1. Calculate det(P) by hand for the matrix P (give details). Compare your answer with the value of `sign` that MATLAB has calculated.
2. Find det(L) (answer without MATLAB, and then check by MATLAB).
3. Calculate det(U).
4. Check (by MATLAB) that $PA = LU$. Explain how this allows you to calculate det(A) from the answers to 1., 2., and 3. Check by using MATLAB.

Question 4. Geometry and Matrices

This question uses MATLAB to illustrate the geometric meaning of some special types of matrices. At the MATLAB prompt type

```
H = house; plot2d(H), hold on
```

A graphics window should open and display a crude drawing of a house. The matrix `H` contains the coordinates of the endpoints of the line segments making up the drawing.

(a) Rotations: Generate a matrix Q by

```
t = pi/6; Q = [cos(t), -sin(t); sin(t), cos(t)]
```

Let Q act on the house by `plot2d(Q*H)` . How has the house been changed? Calculate det(Q). What does this tell you about the area inside the transformed house? (see page 181 of the text). Repeat this with `t = - pi/3` (use $\uparrow$ to save typing) and describe the result. Print the result with the three house images on the same figure.

(b) Dilations: Clear the graphics window and generate a new plot of the house as above. Generate a matrix D by

```
r = .9; D = [r, 0; 0, 1/r ]
```

Let D act on the house by `plot2d(D*H)`. How has the house been changed? Calculate det(D). What does this tell you about the area inside the transformed house? Repeat this with `r = .8` and describe the result. Print the result with the three house images on the same figure.

(c) Shearing Transformations: Clear the graphics window and generate a new plot of the house as above. Generate a matrix T by

```
t = 1/2; T = [1, t; 0, 1]
```

Now let T act on the house by `plot2d(T*H)`. How has the house been changed? Calculate det(T). What does this tell you about the area inside the transformed house? Repeat this with `t = - 1/2` and describe the result. What is the relation between these two transformations? Print the graph with the three house images on the same figure.

Optional Extra-Credit Question: LU vs. RREF for solving $A\mathbf{x} = \mathbf{b}$

In this question you will compare the speed of two methods of solving the equation $A\mathbf{x} = \mathbf{b}$ when A is an invertible square matrix. You will measure the execution time of MATLAB commands using the internal computer clock and the MATLAB function `cputime`.

(a) Write an m-file called `ludata.m` to generate a random 100×100 matrix A, a vector $\mathbf{b} \in \mathcal{R}^{100}$, and calculate the LU decomposition of A by

```
A = rand(100) ;   b = rand(100,1); [L U] = lu(A);
```

Important: Be sure to use the semicolon ; after each command so that these matrices and vector are *not* displayed or included in your diary file.

Now type `ludata`. MATLAB will carry out the operations, but not show any results on the screen.

(b) You can solve $A\mathbf{x} = \mathbf{b}$ by using RREF (Gaussian elimination). The last column $\mathbf{y}$ of the augmented matrix $R = \mathrm{rref}([A\ \mathbf{b}])$ satisfies $A\mathbf{y} = \mathbf{b}$ because `rref(A)` is the identity matrix if A is a random square matrix. Write an m-file called `rrefmeth.m` with the commands

```
t = cputime; R = rref([A b]); y = R(:,101); rreftime = cputime - t
```

Important: Be sure to use the semicolon ; after the first three commands so that the matrix and vector are not displayed or printed in your diary file.

Now type `rrefmeth` to find the CPU time using RREF.

(c) You can also solve $A\mathbf{x} = \mathbf{b}$ by using the LU decomposition of A. Write an m-file called `lumeth.m` with the commands

```
t = cputime; c = inv(L)*b; x = inv(U)*c; lutime = cputime - t
```

Important: Be sure to use the semicolon ; after the first three commands so that the vectors are not displayed or printed in your diary file.

Now type `lumeth` to find the CPU time using the LU method.

(d) Compare the methods **(b)** and **(c)** for speed. Which method of solution was faster? Calculate the ratio rreftime/lutime of the computation times.

(e) According to the table on page 143 of the text, the computation time for Gaussian elimination is approximately $2cn^3/3$, while the time for the LU method (after the L, U factors are already calculated) is approximately $2cn^2$. Here c is a constant depending on the speed of the CPU in the computer and n is the number of equations. Compare the ratio rreftime/lutime that you observed with the predicted ratio for $n = 100$.

Final Editing of Lab Write-up:

After you have worked through all the parts of the lab assignment, edit your diary file. Include the MATLAB calculations, but remove errors that you made in entering commands and remove other material that is not directly related to the questions. If you do the extra-credit question, be sure that you do not print out any of the 100×100 matrices from that questions.