

LAB 4: Vector Spaces and Approximate Solutions to $Ax = b$

In this lab you will use MATLAB to study four topics:

- 1) The subspace spanned by a set of vectors, and the important concepts of *independence*, *basis* and *dimension*.
- 2) The four subspaces associated with an $m \times n$ matrix A : the *column space* $\mathbf{C}(A) \subset \mathbf{R}^m$, the *nullspace* $\mathbf{N}(A) \subset \mathbf{R}^n$, the *row space* $\mathbf{C}(A^T) \subset \mathbf{R}^n$, and the *left nullspace* $\mathbf{N}(A^T)$.
- 3) The best approximate solution to an overdetermined system $Ax = b$ when A has independent columns.
- 4) The method of *least squares* to fit a line to data that have a linear trend and random noise.

Reading from Textbook: Before beginning the Lab, read through Sections 3.5, 3.6, 4.1, 4.2, and 4.3 of the text and work the suggested problems for each section.

Tcodes: For this lab you will need the Teaching Codes

```
fourbase.m, linefit.m, lsq.m,
```

Before beginning work on the Lab questions you should copy these codes from the M.I.T. Teaching Codes directory on the Math Department/Course Materials/Linear Algebra 250 web page to your diskette (see Lab 2 for more details). You will also need the Teaching Codes

```
elim.m, nulbasis.m
```

which should already be on your diskette from Lab 3.

Script files: Before beginning work on the Lab questions, use the text editor in MATLAB to create the following MATLAB script files:

- (a) `rvect.m`: Create a *function* m-file with the commands

```
function v = rvect(m)
v = fix(10*rand(m,1));
```

(note the semicolon on the end of the second line). Save this file under the name `rvect.m` (be sure that you have set the *Path* as described in Lab 2 so that MATLAB can find this m-file). Now test the file by clicking on the MATLAB window and typing

```
v = rvect(3)
```

at the MATLAB prompt. You should get a column vector $\mathbf{v} \in \mathbf{R}^3$ with entries that are (random) integers between 0 and 9. Now type

```
u = rvect(3)
```

You will get another random column vector $\mathbf{u} \in \mathbf{R}^3$. Type `v` at the prompt. You should get the *same* vector \mathbf{v} as before. Note that the name `v` in the `rvect` function file is a *local variable*; you can assign any name to the output. If you have already defined a vector `v` in your work space, it is not changed when you generate `u` by `rvect`.

- (b) `spaces.m`: Write a script file containing the commands

```
B = rmat(2,2); C = rmat(2,3); A = [B, 5*B, C; B, 5*B, C]
R = rref(A)
[ROW, N, COL, LN] = fourbase(A)
```

This uses the script file `rmat.m` that you created in Lab 3, which should already be in your directory of m-files. Save this file with the name `spaces.m`. Test it by typing `spaces` at the MATLAB prompt. You should get a 4×7 matrix A , its row reduced echelon form R , and four matrices `ROW`, `N`, `COL`, `LN` of various sizes. Be sure that you have the Teaching Codes `nulbasis.m`, `fourbase.m` available to MATLAB before continuing.

Lab Write-up: You should open a diary file at the beginning of each MATLAB session (see Lab 1 for details). Begin the diary file with the comment line

```
% Math 250 MATLAB Lab Assignment #4
```

Type `format compact` so that your diary file will not have unnecessary spaces. Put labels to mark the beginning of your work on each part of each question. For example,

```
% Question 1 (a) ...
      :
% Question 1 (b) ...
```

and so on.

Be sure to answer all the questions in the lab assignment. Insert comments in your diary file as you work through the assignment. After you have worked through all the parts of a lab, you will need to edit your diary file, as you did for the previous Labs.

Remove all errors and other material that is not directly related to the questions. For example, remove all material associated with writing and testing your script files.

Preview the document before printing and remove unnecessary page breaks and blank space. Put your name, section number, and student ID number on each page. (If you have difficulty doing this using your text editor, you can write this information by hand after printing the report.)

Important: An unedited diary file without comments will get a GRADE OF ZERO as a lab write up.

Question 1. Independence, Basis and Dimension

Generate four random vectors in \mathbf{R}^3 by the command

```
x = rvect(3), y = rvect(3), z = rvect(3), w = rvect(3)
```

Use these vectors in the following.

(a): Form the matrix A with the vectors \mathbf{x}, \mathbf{y} as columns and calculate its rank.

```
A = [x, y],    r = rank(A)
```

What value for r tells you that the pair \mathbf{x}, \mathbf{y} is linearly independent? Check by calculating a basis for the nullspace of A :

```
N = nulbasis(A)
```

What answer is needed to prove linear independence of \mathbf{x}, \mathbf{y} ? What is the dimension of the subspace spanned by \mathbf{x}, \mathbf{y} ?

Take an arbitrary linear combination \mathbf{u} of \mathbf{x} and \mathbf{y} and calculate the rank of the augmented matrix:

```
u = rand(1)*x + rand(1)*y,    s = rank([A, u])
```

What value for s tells you that u is in the subspace spanned by \mathbf{x}, \mathbf{y} ? Use the up-arrow to generate another \mathbf{u} . Do you get the same value for s ?

Now take an arbitrary linear combination \mathbf{v} of \mathbf{x}, \mathbf{z} and calculate the rank of the augmented matrix:

```
v = rand(1)*x + rand(1)*z,    t = rank([A, v])
```

What value for t will tell you that \mathbf{v} is *not* in the subspace spanned by \mathbf{x}, \mathbf{y} ? Use the up-arrow to generate another \mathbf{v} . Do you get the same value for t ?

(b): Form the matrix B with the vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ as columns and calculate its rank.

```
B = [x, y, z],    r = rank(B)
```

What value for r will tell you that the vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are linearly independent? Check by calculating a basis for the nullspace of B :

```
N = nulbasis(B)
```

What answer is needed to prove linear independence of $\mathbf{x}, \mathbf{y}, \mathbf{z}$? What is the subspace spanned by $\mathbf{x}, \mathbf{y}, \mathbf{z}$?

(c): Form the matrix C with the vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}$ as columns and calculate its rank.

```
C = [x, y, z, w],    r = rank(C)
```

What values for r would tell you that the vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}$ are linearly *dependent*? (see statement **3H** on page 143 of Strang). Check by calculating a basis for the nullspace of C :

```
N = nulbasis(C)
```

Verify that entries in N give a linear relation among the vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}$ by calculating

```
N(1)*x + N(2)*y + N(3)*z + N(4)*w
```

Use this relation to find scalars a, b, c so that

$$\mathbf{x} = a\mathbf{y} + b\mathbf{z} + c\mathbf{w}$$

Use MATLAB to make the calculations and to verify that the linear combination $a\mathbf{y} + b\mathbf{z} + c\mathbf{w}$ that you create does give the vector \mathbf{x} .

Question 2. The Four Subspaces

Generate a random 4×7 matrix A and the associated matrices R , ROW , N , COL , LN by the script file `spaces` that you wrote at the beginning of the lab assignment. Use these matrices in the following.

(a) Column Space: Which columns of R are the pivot columns? How is the matrix COL obtained from A ? Use this information about R to say what the rank of A is. Then check by using MATLAB to calculate `rank(A)`. It is the same as the number of columns of A ? Now calculate `rank(COL)`. Why is it the same as the number of columns of COL ? Finally, calculate

```
rank([COL, A])
```

Explain why this is the same as the number of columns of `rank(COL)` .

(b) Left Null Space: The columns of LN are the *special solutions* to $A^T y = 0$. These column vectors are orthogonal to the columns of A . Check this by calculating $A' * LN$.

The *Fundamental Theorem of Linear Algebra* says that

$$\# \text{ columns of COL} + \# \text{ columns of LN} = ?$$

(fill in the question mark). Does this agree with your MATLAB results?

(c) Row Space: The columns of ROW are a basis for the column space of A^T . How are they obtained from the matrix R^T ? Are these columns the *same vectors* as any of the columns of A^T ? Calculate

```
rank([ROW, A'])
```

Explain why this is the same as the number of columns of ROW.

(d) Null Space: The columns of N are the *special solutions* to $Ax = 0$. These column vectors are orthogonal to the columns of A^T . Check this by calculating $ROW' * N$.

The *Fundamental Theorem of Linear Algebra* says that

$$\# \text{ columns of ROW} + \# \text{ columns of N} = ?$$

(fill in the question mark). Does this agree with your MATLAB results?

Question 3. Orthogonal Projections and Approximate Solution to $Ax = b$

(a) Projection onto a Subspace: Generate two random vectors $\mathbf{y}, \mathbf{z} \in \mathbf{R}^4$ and form the matrix A with these vectors as columns.

```
y = rvect(4), z = rvect(4), A = [y, z]
```

What value for `rank(A)` is needed to show that \mathbf{y}, \mathbf{z} is a linearly independent set? Verify by MATLAB that this value is obtained. Let V be the subspace of \mathbf{R}^4 spanned by \mathbf{y}, \mathbf{z} . What is $\dim(V)$?

Calculate $A' * A$. What value for `rank(A' * A)` is needed to show that the matrix $A^T A$ has an inverse? Verify by MATLAB that this value is obtained. Then calculate the matrix

```
P = A * inv(A' * A) * A'
```

This is the orthogonal projection matrix onto V . Observe that $P = P^T$. Calculate $P * \mathbf{y}$, $P * \mathbf{z}$. Redisplay \mathbf{y} , \mathbf{z} and observe that $P\mathbf{y} = \mathbf{y}$ and $P\mathbf{z} = \mathbf{z}$.

(b) Orthogonal Decomposition $\mathbf{b} = \mathbf{p} + \mathbf{e}$: Generate another random vector $\mathbf{b} = \text{rvect}(4)$ and consider the questions

(i) Is $\mathbf{b} \in V$?

(ii) Is $P\mathbf{b} \in V$?

First answer these questions by calculating `rank([y, z, b])` and `rank([y, z, P*b])` (remember Question #1). Now answer these questions in another way by calculating $P * \mathbf{b}$, $P * P * \mathbf{b}$. Explain how these calculations give answers to questions (i) and (ii).

Finally, set $\mathbf{p} = P * \mathbf{b}$ and $\mathbf{e} = \mathbf{b} - \mathbf{p}$. Check by MATLAB that

$$\mathbf{p}' * \mathbf{e} = 0, \quad \mathbf{y}' * \mathbf{e} = 0, \quad \mathbf{z}' * \mathbf{e} = 0$$

(due to roundoff error, the answers will be very small numbers on the order of 10^{-14}). So \mathbf{p} is in the subspace V and \mathbf{e} is the component of \mathbf{b} that is perpendicular to the subspace V (see Figure 4.5 on page 179 of Strang).

(c) Approximate Solution to $A\mathbf{x} = \mathbf{b}$: Take the 4×2 matrix $A = [\mathbf{y}, \mathbf{z}]$ as in part (a) and the vector $\mathbf{b} \in \mathbf{R}^4$ as in part (b). Explain how the results of part (b) show that

(i) There is **no** vector $\mathbf{x} \in \mathbf{R}^4$ that satisfies the equation $A\mathbf{x} = \mathbf{b}$

(ii) There is a vector $\mathbf{x} \in \mathbf{R}^4$ that satisfies the modified equation $A\mathbf{x} = P\mathbf{b}$, where P is the projection matrix from part (a).

Calculate the *Least-Squares Approximate Solution* in (ii) by

$$\mathbf{x} = \text{inv}(\mathbf{A}' * \mathbf{A}) * \mathbf{A}' * \mathbf{b}$$

Check by MATLAB that $\mathbf{A} * \mathbf{x} = \mathbf{p}$, where $\mathbf{p} = P\mathbf{b}$ is the projection of \mathbf{b} onto the column space of A .

Question 4. Fitting a Line to Data Points

(a) Generating and Plotting Linear Data: Define a column vector of ten equally-spaced t values

$$\mathbf{t} = [1:10]'$$

Now generate a column vector of the corresponding values of the linear function $y = 4 + t$ and plot it by

$$\mathbf{y} = 4 + \mathbf{t}; \text{linefit}(\mathbf{t}, \mathbf{y})$$

(Note the semicolon that suppresses the display of the y data). MATLAB should open a new window in which this line is plotted. Notice that the line passes exactly through each data point, and the 'best fit' equation is the given function $4 + t$. Print this graph by clicking on **file** in the figure tool bar, and then click on **print**. Include the printed copy in your lab write-up.

(b) Linear Data with Random Noise: Now add some random noise to each y data value in part (a):

$$\mathbf{y} = 2 + 4 * \text{rand}(10, 1) + \mathbf{t}; \text{linefit}(\mathbf{t}, \mathbf{y})$$

These random data points don't lie on the line from part (a), even though they show the same general trend. The line that is plotted is the *best fitting* line; it minimizes the *mean square error* between the y coordinates on the line and the data values. Notice how some data points are above the line, and others are below the line. The equation of the best line fitting the random data points is displayed above the graph. Print this graph (following the same procedure as in part (a)). Include the printed copy in your lab write-up.