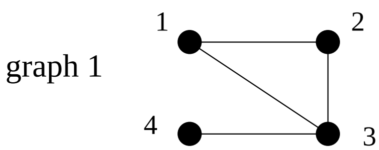
**Intro. to Linear Algebra 250-C**
**Extra Credit Project 1 — Graphs and Matrices**

Please write all answers on separate sheets of paper. Your answers should be *numbered* and in the *same order* in which the problems appear. Your project should be *stapled* and your name should appear on *every* sheet.

**Introduction:** In this project you will use linear algebra to learn about *graphs*. A graph is a figure such as:



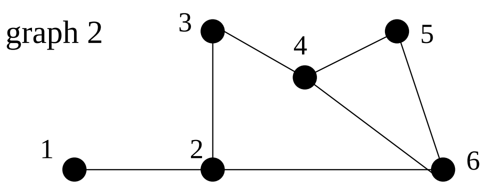

The numbered dots represent *vertices*. These vertices are joined by *edges*, lines which connect one vertex to another vertex. To each graph, we can associate a special type of matrix called the *adjacency matrix*. It is defined as follows. Suppose that the vertices of the graph are numbered 1 to $n$. Then each component of the adjacency matrix $(a_{ij})$ is defined by

$$a_{ij} = \begin{cases} 1 & \text{if there is an edge connecting } i \text{ and } j \\ 0 & \text{otherwise} \end{cases}$$

The above graph has the following adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

1. Find the adjacency matrix $A$ for the following graph.




2. Notice that $A$ is symmetric. Do you think that the adjacency matrix of every graph is symmetric? If yes, give a reason. If not, give an example of a graph whose adjacency matrix is not symmetric.

We say that an edge has length 1. A path connecting two vertices has length $k$ if it consists of $k$ edges. For example, the path

$$1 \rightarrow 2 \rightarrow 3 \rightarrow 4$$

in graph #2 has length 3. The following path has length 2.

$$1 \to 2 \to 1$$

**Theorem:** *(Interpretation of the powers of an adjacency matrix). If $A$ is the adjacency matrix of a graph, then the $(i, j)$ entry of $A^k$ is a nonnegative integer which represents the number of paths of length $k$ from vertex $i$ to vertex $j$ in the graph.*

In order to understand why this theorem is true, examine the $(6, 3)$ entry of $A^2$. Using the "Row-Column Rule", the $(6, 3)$ entry in $A^2$ looks like $a_{61}a_{13}+a_{62}a_{23}+a_{63}a_{33}+a_{64}a_{43}+a_{65}a_{53}+a_{66}a_{63}$.

3. Evaluate each term in $a_{61}a_{13} + a_{62}a_{23} + a_{63}a_{33} + a_{64}a_{43} + a_{65}a_{53} + a_{66}a_{63}$. The first two summands $a_{61}a_{13} + a_{62}a_{23}$ are evaluated for you.

$$(0)(0) + (1)(1)+\underline{\hspace{6cm}} = \underline{\hspace{1cm}}$$

4. Explain what each term in the above sum tells us about paths of length 2 from vertex 6 to vertex 3. (For example, $a_{62}a_{23} = (1)(1) = 1$. This says that the length one paths $6 \to 2$ and $2 \to 3$ appear in the graph and together they give one path of length two from vertex 6 to vertex 3.)

5. Use MATLAB to calculate $A^2$ and $A^3$. Record your results.

6. Are there pairs of vertices $i$ and $j$ that are not connected by a path of length two but are connected by a path of length three? What are these pairs?

7. How many paths of length two are there from vertex 3 to itself? What are they?

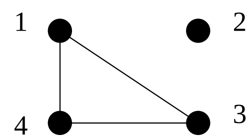
Now use MATLAB to compute the matrix $B = A + A^2 + A^3$.

8. Are there any zero entries in $B$? What does that represent in terms of paths?

Graphs #1 and #2 are *connected*. That means that, for any two pairs of vertices, you can find a path connecting the two vertices.

An example of a graph that is *not* connected is:

graph 3




Notice that there is no edge connecting vertex 2 with any other vertices.

9. Determine the adjacency matrix $C$ for this graph. What are the components in the second row and second column of $C$? How do they tell you that there are no paths connecting vertex 2 with other vertices?

10. Using MATLAB, calculate $C^2$, $C^3$, $C^4$ and $C^5$. What do you notice about the second row and column in each of these matrices? Do you think that this holds for all powers of $C$?

11. Using the "Row-Column Rule" and induction on $k$, prove your conjecture from problem 10. for $C^k$ for all natural numbers $k \geq 1$.

**Application:** An airline's flight routes can be represented by a graph. Each airport is represented by a vertex and each nonstop flight route is represented by an edge between the two airports. We will assume that planes fly in both directions between airports. Suppose that a small airline has the following nonstop flight routes:

| Airport 1: | Airport 2: |
|---|---|
| Newark | Los Angeles |
| Newark | Dallas |
| Newark | Orlando |
| Los Angeles | Seattle |
| Los Angeles | Dallas |
| Dallas | Seattle |
| Orlando | Miami |

12. Number the cities as follows: (1) Los Angeles, (2) Newark, (3) Orlando, (4) Dallas, (5) Seattle, (6) Miami. Draw the graph that represents the above set of nonstop flight routes.

13. Find the adjacency matrix $A$ of this graph and enter it into MATLAB.

14. Use MATLAB to compute the sums $A + A^2$, $A + A^2 + A^3$, ... (you figure out when to stop). What is the largest number of flight segments necessary to fly between any two of the five cities? Explain your answer.

15. Why are the nonstop flights between Orlando and Newark important to this airline? (Hint: Look at what happens in 12. if you remove the edge).

16. Can you add just one nonstop flight to the above schedule so that passengers can fly between any two of the five cities using at most two flight segments? If that is not possible, try adding more flights until at most two flight segments are needed. Try to add as few as possible and explain your process.