Note: A. This problem set concentrates on material from the end of the course. **For a complete review, you should also study the review problem sets for the two in-class exams. Please consider these earlier problem sets as implicitly included with this one.** Particular topics that should be reviewed from earlier sets include: (i) Solving systems of linear equations, row operations, elementary matrices; (ii) The LU decomposition of a matrix; (iii) Inverses of matrices; (iv) Subspaces, finding bases for Col A, Row A, and Null A; (v) Determinants.

B. For section 250:C2: You should know the precise definitions of: a *subspace* of a vector space, *linear independence* of a sequence of vectors, the space *spanned* by a set of vectors, a *basis* of a vector space, *eigenvalues and eigenvectors.*

1. Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 3 \end{bmatrix}$.

(a) Find an orthonormal set of eigenvectors of A which form a basis for $\mathbb{R}^3$. Hint: $t = -1$ is a root of the characteristic polynomial.

(b) Find an orthonormal matrix Q and a diagonal matrix D such that such that $A = QDQ^T$.

2. Suppose that A is a symmetric $n \times n$ matrix and that $A\mathbf{x} = 4\mathbf{x}$ and $A\mathbf{v} = -4\mathbf{v}$ for certain nonzero vectors $\mathbf{x}, \mathbf{v} \in \mathbb{R}^n$. Show that $\mathbf{x}$ and $\mathbf{v}$ are orthogonal.

3. State the Cauchy-Schwarz inequality and the triangle inequality, and show that the former implies the latter.

4. Given that $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$, and that $\mathbf{w}$ is a vector in $\mathbb{R}^3$ with $\|\mathbf{w}\| = 5$, $\mathbf{w} \cdot \mathbf{u} = 13$, and $\mathbf{w} \cdot \mathbf{v} = -5$, compute $\mathbf{u} \cdot \mathbf{v}$ and $(\mathbf{u} + 2\mathbf{w}) \cdot (\mathbf{u} - 3\mathbf{v} - \mathbf{w})$.

5. True or false; justify your answers. A 4×4 matrix A is *necessarily* diagonalizable if:

T F (a) A is symmetric;

T F (b) A has four linearly independent eigenvectors;

T F (c) The eigenvalues of A are 7, -2, and 0, and A has rank 2;

T F (d) The characteristic polynomial of A is $(\lambda^2 - 1)(\lambda^2 - 2)$;

6. (a) Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$.

(b) Find a matrix P and diagonal matrix D such that $D = P^{-1}AP$.

7. A certain 3×3 matrix A has eigenvalues $\lambda_1 = 2$, $\lambda_2 = 1$, and $\lambda_3 = -1$, and corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(a) Use the formula $A = P^{-1}DP$ (for suitable P and D) to find A.

(b) Let $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$. Use (a) to find coefficients c_1, c_2, c_3 so that $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$. Then compute $A^n\mathbf{x}$ from this formula for $\mathbf{x}$ for arbitrary $n > 0$. What is a good approximation to $A^n\mathbf{x}$ for n large?

8. Find a 3×3 orthogonal matrix Q with first column proportional to $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

9. Let $A = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$.

(a) Give the dimensions of $\operatorname{Row} A$, $\operatorname{Col} A$, $\operatorname{Null} A$, and $\operatorname{Null} A^T$.

(b) Find orthonormal bases for $\operatorname{Row} A$, $\operatorname{Col} A$, and $\operatorname{Null} A$. Hint: one of these is a trivial problem requiring no calculation and one requires almost no calculation. Which are these?

10. Classify each statement as true (T) or false (F). If your answer if T, **give a brief proof** showing that the statement is *always* true; if your answer is F, **give a specific example** for which the statement is not true.

T F (a) If Q is an orthogonal matrix then the eigenvalues of Q are real.

T F (b) The null space of a matrix A is the orthogonal complement of the column space of A.

T F (c) Every orthogonal matrix has null space $\{\mathbf{0}\}$.

T F (d) If P and Q are orthogonal matrices then P^TQ is orthogonal.

T F (e) if $\mathcal{S}$ is a subset of $\mathbb{R}^n$ then $(\mathcal{S}^\perp)^\perp = \mathcal{S}$.

T F (f) If A is an $n \times n$ matrix and 0 is an eigenvalue of A then $\operatorname{Row} A \neq \mathbb{R}^n$.

T F (g) If Q is an orthogonal matrix then $Q = Q^{-1}$.

T F (h) If A is any nonzero matrix then A^TA is invertible.

T F (i) If Q is an orthogonal matrix then $\operatorname{Col}(Q^T) = \operatorname{Col} Q$.

T F (j) If A is an $n \times n$ matrix then eigenvectors for distinct eigenvalues of A are orthogonal.

11. Suppose that W is a subspace of $\mathbb{R}^n$ of dimension k and that $\mathbf{w}_1, \ldots, \mathbf{w}_k, \mathbf{w}_{k+1}, \ldots, \mathbf{w}_n$ is an orthonormal basis for $\mathbb{R}^n$ such that $\mathbf{w}_1, \ldots, \mathbf{w}_k$ is a basis for W.

(a) Any vector $\mathbf{u} \in \mathbb{R}^n$ has an expansion $\mathbf{u} = c_1\mathbf{w}_1 + \cdots + c_n\mathbf{w}_n$. Give a simple formula for the coefficients c_j.

(b) We know that any $\mathbf{u} \in \mathbb{R}^n$ can be written uniquely as $\mathbf{u} = \mathbf{w} + \mathbf{z}$, with $\mathbf{w} \in W$ and $\mathbf{z} \in W^\perp$. Explain why $\mathbf{w} = c_1\mathbf{w}_1 + \cdots + c_k\mathbf{w}_k$.

(c) Let C be the $n \times k$ matrix with columns $\mathbf{w}_1, \ldots, \mathbf{w}_k$. *Using your answers to (a) and (b)*, show that P_W, the orthogonal projection matrix onto W, is given by $W = CC^T$. (Recall that , in the notation of (b), $\mathbf{w} = P_W\mathbf{u}$.)

(d) Show that $C^TC = I_k$; using this, derive the result in (c) from the general formula for P_W.

12. Consider the data points $(-3, 9)$, $(-1, 7)$, $(0, 5)$, $(4, 1)$.

(a) The method of least squares for a straight line fit to this data minimizes a certain quantity. What is that quantity in this case? Give the answer explicitly; define any variables used.

(b) We obtain a solution by solving the *normal equations*; in the notation used in the text these are $C^TC\mathbf{x} = C^T\mathbf{y}$. What is C for the data above? What is $\mathbf{y}$? What is $\mathbf{x}$?

(c) Find the equation of the straight line which best fits this data.