1. In each part below give the precise definition in one or more full sentences.
(a) What does it mean for a set of vectors \( S = \{ \mathbf{u}_1, \ldots, \mathbf{u}_k \} \) to be linearly independent?

In (b)–(f) define the given term:
(b) The span of a set of vectors \( S = \{ \mathbf{u}_1, \ldots, \mathbf{u}_k \} \);
(c) A subspace of \( \mathbb{R}^n \);
(d) A basis of a subspace \( W \) of \( \mathbb{R}^n \);
(e) The dimension of a subspace \( W \) of \( \mathbb{R}^n \).
(f) An eigenvector and corresponding eigenvalue of a square matrix \( A \).

2. (a) Suppose that \( A \) is an \( m \times n \) matrix. Define the null space \( W \) of \( A \). For what value of \( k \) is \( W \) a subset of \( \mathbb{R}^k \)?
(b) Show that \( W \) is in fact a subspace of \( \mathbb{R}^k \) (with \( k \) as in (a)) by checking all the conditions in the definition of a subspace.

3. Suppose that \( S = \{ \mathbf{u}_1, \ldots, \mathbf{u}_k \} \) is a finite set of vectors in \( \mathbb{R}^n \). Show that \( \text{Span} \, S \) is a subspace of \( \mathbb{R}^n \) by checking all the conditions in the definition of a subspace.

4. Find the \( A = LU \) factorization (that is, find \( L \) and \( U \)) of \( A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 2 & 2 & -2 & 3 \\ 0 & 3 & 2 & 7 \end{bmatrix} \) Then use it to solve \( Ax = [3 \ -4 \ 10 \ 12]^T \) by solving two equations: one with \( L \) and then one with \( U \).
(b) Give an example of a \( 4 \times 4 \) invertible matrix which does not have an \( LU \) factorization.

5. Find a basis for \( \text{Span} \, (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4) \), where

\[
\mathbf{w}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} 2 \\ -3 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -5 \end{bmatrix}, \quad \mathbf{w}_4 = \begin{bmatrix} 2 \\ -2 \\ 3 \\ 0 \end{bmatrix}.
\]

6. Given that \( A \) has reduced row echelon form \( R \), where

\[
A = \begin{bmatrix} 2 & -4 & 2 & -2 \\ 2 & -4 & 3 & -4 \\ 4 & -8 & 3 & -2 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},
\]

find bases for the row space, null space, and column space of \( A \). Give the rank and nullity of \( A \) and of \( A^T \).

7. Classify each statement as true or false and give a brief justification of your answer.
(a) If \( Ax = 0 \) has a unique solution, then there are no vectors in the nullspace of \( A \).
(b) If \( \mathbf{u} \) and \( \mathbf{v} \) belong to a subspace \( W \) of \( \mathbb{R}^n \) and \( c \) and \( d \) are scalars, then \( c \mathbf{u} + d \mathbf{v} \) belongs to \( W \).
(c) A square matrix \( A \) is invertible if and only if \( \det A = 0 \).
(d) If \( A \) is an \( m \times n \) matrix and \( n > m \), then the nullspace of \( A \) is not \( \{0\} \).
(e) If \( A \) is an \( m \times n \) matrix, then \( \text{dim} \, \text{Null} \, A + \text{dim} \, \text{Col} \, A = n \).
(f) The nullity of a matrix \( A \) is always equal to the nullity of \( A^T \).
(g) If \( A \) is an \( n \times n \) matrix and rank \( A < n \), then 0 is a root of the characteristic polynomial of \( A \).
(h) If \( \lambda \) is an eigenvalue of \( A \) with algebraic multiplicity \( r \) and \( W \) is the corresponding eigenspace, then \( \dim W = r \).
(i) Every \( n \times n \) matrix with \( n \) distinct eigenvalues is diagonalizable.
8. Determine which of the following are subspaces of $\mathbb{R}^3$. For each subspace give its dimension. Justify your answers.

(a) Span \(\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ 0 \end{bmatrix} \right\} \); 
(b) \(\left\{ \begin{bmatrix} r \\ -8s \\ r + s \end{bmatrix} : r, s \in \mathbb{R} \right\} \); 
(c) \(\left\{ \begin{bmatrix} r + s \\ -8(r + s) \\ 2r + 2s \end{bmatrix} : r, s \in \mathbb{R} \right\} \); 
(d) \(\left\{ \begin{bmatrix} r \\ -8s \\ r + s + 1 \end{bmatrix} : r, s \in \mathbb{R} \right\} \); 
(e) \(\left\{ \begin{bmatrix} r \\ s \\ t \end{bmatrix} : r, s, t \in \mathbb{R} \right\} \); 
(f) \(\left\{ \begin{bmatrix} r \\ -8r \\ 2r \end{bmatrix} : r = 0 \right\} \).

9. (a) Evaluate the determinant \(\begin{vmatrix} 1 & -2 & 5 \\ -3 & 4 & 2 \\ 3 & -6 & -2 \end{vmatrix}\) by a cofactor expansion along the second row.
(b) Evaluate the determinant \(\begin{vmatrix} 1 & -1 & 0 \\ 2 & 1 & 3 \\ 1 & -2 & 1 \\ 3 & -3 & 2 \end{vmatrix}\) by reducing the matrix to triangular form and keeping track of the row operations used.

10. Let \(A = \begin{bmatrix} a \\ b \\ c \end{bmatrix}\) be a \(3 \times 3\) matrix with row vectors \(\mathbf{a}\), \(\mathbf{b}\), and \(\mathbf{c}\). Assume that \(\det A = 5\). Find:

(a) the determinant of the matrix \(\begin{bmatrix} \mathbf{c} + \mathbf{b} \\ \mathbf{a} + 2\mathbf{b} \\ \mathbf{a} - \mathbf{b} - \mathbf{c} \end{bmatrix}\);

(b) the determinant of the matrix \(AC^2AC\), where \(C = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & -2 \end{bmatrix}\).

11. Let \(\mathbf{v}\) be a nonzero vector in \(\mathbb{R}^4\), and let \(A = \mathbf{vv}^T\). (\(A\) is a \(4 \times 4\) matrix.)

(a) Show that \(\mathbf{v}\) is an eigenvector of \(A\). What is the eigenvalue? (Hint: compute \(A\mathbf{v}\) using the associative property of matrix multiplication.)

(b) What is the rank of \(A\)? What is \(\dim \text{Null} A\)? What are all the eigenvalues of \(A\), with their multiplicities?

(c) If \(A = PDP^{-1}\) with \(D\) diagonal, what must \(D\) be?

12. (a) Find the eigenvalues and eigenvectors of the matrix \(A = \begin{bmatrix} 2 & -1 \\ 4 & -3 \end{bmatrix}\).

(b) Find an invertible matrix \(P\) and diagonal matrix \(D\) such that \(A = PDP^{-1}\).

13. The matrix \(A = \begin{bmatrix} 2 & -3 & 2 \\ -1 & -6 & 9 \\ -5 & -1 & 5 \end{bmatrix}\) has characteristic polynomial \(-(t + 2)^2(t - 5)\).

(a) Find the eigenvalues of \(A\) and the multiplicities of each eigenvalue.

(b) For each eigenvalue found above give a basis for the corresponding eigenspace.

(c) Determine whether or not \(A\) is diagonalizable.