1. (a) What does it mean to say that a set \( S = \{u_1, \ldots, u_k\} \) of vectors in \( \mathbb{R}^n \) is linearly independent? Give the precise definition in one or more full sentences. Then describe what this means in terms of the matrix \( A \) with columns \( u_1, \ldots, u_k \).

(b) Do the three vectors \( u_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 3 \\ 5 \\ -3 \end{bmatrix}, \quad \text{and} \quad u_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \) make up a linearly independent set? Justify your answer using the matrix \( A \) as in (a).

2. (a) What is meant by the span of a set \( S = \{u_1, \ldots, u_k\} \) of vectors in \( \mathbb{R}^n \)? Give the precise definition in one or more full sentences. Then describe what this means in terms of the matrix \( A \) with columns \( u_1, \ldots, u_k \).

(b) Suppose that \( u_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 3 \\ 5 \\ -3 \end{bmatrix}, \quad \text{and} \quad u_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \) is the span of the set of vectors \( S = \{u_1, u_2, u_3\} \) all \( \mathbb{R}^3 \)? Justify your answer using the matrix \( A \) as in (a).

3. Answer questions 1 and 2 when \( u_3 \) is changed to \( u_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \).

4. In (a)-(c) below we suppose that we have a system of equations \( Ax = b \) and that we have used row operations to transform the augmented matrix \( [A \ b] \) to the reduced row-echelon form \( [R \ c] \) given below. In each case, determine (i) whether the original equations have a solution; (ii) if they do have a solution, whether or not it is unique; and (iii) if it is not unique, how many free parameters there are in the solution. Then write the solution explicitly as a fixed vector plus a linear combination of vectors \( y \) that satisfy \( Ay = 0 \) with the free variables as coefficients.

(a) \[
\begin{bmatrix}
1 & 5 & 0 & 2 & 8 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
1 & 0 & 0 & 0 & 2 \\
0 & 1 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 3 \\
0 & 0 & 0 & 1 & 4
\end{bmatrix}
\]

(c) \[
\begin{bmatrix}
0 & 1 & 2 & 0 & -2 & 0 & 2 \\
0 & 0 & 0 & 1 & 3 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

5. In each part below, give a \( m \times n \) matrix \( R \) in reduced row-echelon form satisfying the given condition, or explain briefly why it is impossible to do so.

(a) \( m = 3, \ n = 4 \), and the equation \( Rx = c \) has a solution for all \( c \).

(b) \( m = 3, \ n = 4 \), and the equation \( Rx = 0 \) has a unique solution.

(c) \( m = 4, \ n = 3 \), and the equation \( Rx = c \) has a solution for all \( c \).

(d) \( m = 4, \ n = 3 \), and the equation \( Rx = 0 \) has a unique solution.

(e) \( m = 4, \ n = 4 \), and the equation \( Rx = 0 \) has no solution.

(f) \( m = 4, \ n = 4 \), and the equation \( Rx = 0 \) has a nontrivial solution.

(g) \( m = 4, \ n = 4 \), and for every \( c \) the equations \( Rx = c \) have a solution containing a free parameter.
6. (a) Suppose that \( u \) and \( v \) are solutions of the system of equations \( Ax = 0 \). Show that \( cu + dv \) is also a solution, for any scalars \( c \) and \( d \).

(b) Why does the above conclusion not hold (in general) if the system of equations is \( Ax = b \) with \( b \) a nonzero vector?

7. Suppose that

\[
A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 & -1 \\ -1 & 2 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -2 & 3 \\ 3 & -3 \\ 4 & 1 \end{bmatrix}.
\]

Which of the following quantities are defined? Calculate those that are defined.

(a) \( BA \)

(b) \( AB \)

(c) \( 3C - 2B^T \)

(d) \( BC \)

(e) \( CAB \)

(f) \( C + 2A \)

(g) \( C^T C \).

8. Let \( A \) be an \( m \times n \) matrix of rank \( r \). What can you conclude about \( m \), \( n \), and \( r \) (other than \( r \leq m \) and \( r \leq n \) which is always true) if the equation \( Ax = b \) has

(a) exactly one solution for some \( b \) and no solution for other \( b \)?

(b) infinitely many solutions for all \( b \)?

(c) exactly one solution for every \( b \)?

(d) infinitely many solutions for some \( b \) and no solutions for other \( b \)?

(e) exactly one solution when \( b = 0 \)?

9. (a) Suppose that \( A \) and \( B \) are \( 4 \times 5 \) matrices and that \( B \) is obtained from \( A \) by the row operation given below. In each case give an elementary matrix \( E \) such that \( B = EA \).

(i) \( r_1 \leftrightarrow r_4 \),

(ii) \( r_3 + 3r_2 \rightarrow r_3 \),

(iii) \( 7r_2 \rightarrow r_2 \).

(b) Give the inverses of the elementary matrices found in (i), (ii), and (iii) above. (You can do this without calculation; think about reversing the corresponding row operations.)

10. A certain \( 3 \times 3 \) matrix \( A = [a_1 \ a_2 \ a_3] \) has reduced row echelon form \( R = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \).

(a) Find a nontrivial linear relation on the columns of \( A \), that is, a relation \( c_1a_1 + c_2a_2 + c_3a_3 = 0 \) with \( c_1 \), \( c_2 \), and \( c_3 \) not all zero.

(b) Suppose that \( a_1 = \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} \) and \( a_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \). Find \( a_3 \).

11. (a) Suppose that \( A \) is a square matrix. What does it mean to say that \( A \) is invertible? (Give the definition, not one of the many equivalent conditions in Theorem 2.6 of the text.)

(b) Suppose that \( A \) and \( B \) are invertible \( n \times n \) matrices. Show that \( (AB)^{-1} = B^{-1}A^{-1} \).

(c) Suppose that \( A \) is an invertible \( n \times n \) matrix. Show that \( (A^T)^{-1} = (A^{-1})^T \).

12. Use row reduction to show that the matrix

\[
\begin{bmatrix} 0 & 2 & -1 \\ 1 & -1 & 2 \\ 2 & -1 & 3 \end{bmatrix}
\]

is invertible and to find its inverse.

13. Do the True-False questions from Sections 1.1–1.4, 1.6, 1.7, 2.1, 2.3, and 2.4 that are listed in the homework assignments.