(12.3) Exercise 63 Interpret and prove

$$|\mathbf{a}+\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2 = 2\,|\mathbf{a}|^2 + 2\,|\mathbf{b}|^2\,.$$

Method: For interpretation, draw a picture starting from general **a** and **b** with other vectors constructed. For proof, carefully use $|\mathbf{v}|^2 = \mathbf{v}\cdot\mathbf{v}$ and linearity of dot product.

Equation of line. A line in space is determined by a point $P_0(x_0, y_0, z_0)$ and a direction $\mathbf{v} = \langle a, b, c\rangle$. Once you have the six numbers, all that is needed to get the equation is to write them in the right places. Things are slightly complicated by three ways of writing the equation being in common use. The keywords identifying these forms will appear on exams without warning.

Vector form. The form closest to the geometry of the line uses the idea that vectors in the same direction are related by being scalar multiples of one another. Thus, the vector $\overrightarrow{P_0P}$ from the given point P_0 to the variable point $P(x, y, z)$ should be a multiple of the vector **v**. The scalar multiple is usually denoted by t, although different letters should be used for different lines in the same problem. Changing the parentheses around the quantities in P_0 and P to angle brackets transforms them into the vectors $\mathbf{r}_0 = \overrightarrow{OP_0}$ and $\mathbf{r} = \overrightarrow{OP}$, and the vector equation becomes

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}.$$

Parametric form. The vector equation is an abbreviation for the three scalar equations obtained by equating the components. For the data given above, this is

$$x = x_0 + at$$
$$y = y_0 + bt$$
$$z = z_0 + ct$$

Symmetric form. Each of these equations can be solved for the parameter t. The point P lies on the line if you always get the same answer. This puts the equation in a form that emphasizes a test for whether P lies on the line:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

This may even be used if some of the coefficients a, b or c are zero. The other forms show that this should be interpreted as requiring that a zero denominator requires the corresponding numerator to be zero.

How else could a line be given? Another way to give a line is by specifying two points P_0 and $P_1(x_1, y_1, z_1)$ on the line. This is immediately converted into the original form by taking $\mathbf{v} = \overrightarrow{P_0P_1}$.

A line may also be given as an intersection of two planes. We will return to this after discussing how planes are given.

Parallel lines; intersections of lines; skew lines. Lines are said to be parallel if they have the same direction. Thus, $P_0 + t\mathbf{v}_0$ is parallel to $P_1 + u\mathbf{v}_1$ if there is a constant λ such that $\mathbf{v}_0 = \lambda\mathbf{v}_1$. The *inclusive* form of this definition means that *identical* lines are considered to be parallel. In our example, the lines are identical if $\overrightarrow{P_0P_1}$, $\mathbf{v}_0$, and $\mathbf{v}_1$ all have the same direction. Although the same line may have two equations that appear different, it is easy to check whether different equations represent the same line.

Two lines intersect if you can find t and u so that

$$P_0 + t\mathbf{v}_0 = P_1 + u\mathbf{v}_1.$$

In scalar form, this gives three equations relating the two variables t and u. This suggests that pairs of lines do not usually intersect. A simple example is

$$t\langle 1, 0, 0\rangle \text{ and } (0, 0, 1) + u\langle 0, 1, 0\rangle.$$

Lines that are not parallel and do not intersect are called *skew*.

Exercises

#3. Find line through $(-2, 4, 10)$ and parallel to $\langle 3, 1, -8\rangle$.

Method. Reformat given numbers.

#7. Find line though points $(3, 1, -1)$ and $(3, 2, -6)$.

Method. First, find direction.

#11. Show that the line through $(2, -1, -5)$ and $(8, 8, 7)$ is parallel to the line through $(4, 2, -6)$ and $(8, 8, 2)$.

Method. Determine directions.

#15. Determine whether

$$\frac{x-4}{2} = \frac{y+5}{4} = \frac{z-1}{-3} \qquad (L_1)$$

$$\frac{x-2}{1} = \frac{y+1}{3} = \frac{z}{2} \qquad (L_2)$$

are parallel, skew or intersecting.

Method. Determine directions.

Vector functions. We consider functions from $\mathbb{R}$ to $\mathbb{R}^3$. The appropriate notion of continuity of such functions is that each component be continuous. Similarly, to differentiate such a function, use the derivatives of the components as the components of a vector. These properties are not arbitrary: they can be proved from a general definition of limit. If the values of a vector function are plotted, one gets a **space curve**. The simplest example is a linear function, whose graph is a line. The **parametric equations** of plane curves considered in Chapter 10 are also special cases of this definition. Several other examples appear in the text. An interesting example is the **helix**, given by $\langle \cos t, \sin t, t \rangle$. On the surface where $x^2 + y^2 = 1$, this curve moves evenly around the cylinder and up the axis. Examples appearing in the exercises for Section 13.1 are

$$x = t \quad y = \frac{1}{1+t^2} \quad z = t^2 \tag{9}$$

$$x = \cos t \quad y = \sin t \quad z = \sin 5t \tag{11}$$

$$x = \sin t \quad y = 3 \quad z = \cos t \tag{17}$$

Derivatives. If $\mathbf{r}(t)$ is a vector function, then

$$\mathbf{r}(t+h) - \mathbf{r}(t)$$

is a difference of vectors for each t and h, so it is the vector whose components are $x(t+h) - x(t)$, $y(t+h) - y(t)$ and $z(t+h) - x(t)$. Dividing the vector difference by the scalar h involves dividing each component by h, but it also admits a geometric description as a rescaling of the vector difference. For reasonable functions $x(t)$, $y(t)$, and $z(t)$, the difference quotients have a limit which is the derivative. This means that the rescaled difference of vectors also has a limit which is the vector of derivatives. We describe the direction of the vector here, and its length will be interpreted in the next section. The difference $\mathbf{r}(t+h) - \mathbf{r}(t)$ is a chord of the curve, and the rescaling simply replaces the chord by a vector in the same direction whose length is (usually) bounded away from zero as $h \to 0$ for fixed t. The limit of these directions is the direction of the limit. As with plane curves, such a limit of chords is what we expect to be a **tangent direction** to the curve at the point $\mathbf{r}(t)$.

In particular, one can scale the direction to obtain the **unit tangent vector T**.

A **tangent line** to the curve at this point is the line through $\mathbf{r}(t)$ in the direction of $\mathbf{r}'(t)$. Usually, a numerical value of t will be given (perhaps indirectly by specifying the value of $\mathbf{r}(t)$) which will be substituted into these expressions (after finding $\mathbf{r}'(t)$ from the function $\mathbf{r}(t)$).

The calculus of this derivative has all of the expected properties: the derivative of a sum is the sum of the derivative and the derivative of a scalar constant multiple is that multiple of the derivative. Also, all of the products that we have met are build in some way from products of a component of the first factor times a component of the second factor. This can be used to prove that all analogs of the product rule hold. There is also a chain rule for $\mathbf{r}(t(u))$.

Exercises 13.2

All exercise are some variation of finding the derivative, sometimes scaled to give **T**. In some problems, the numerical description of a point on the curve is given, allowing numerical values of these quantities and a tangent line and a tangent line to be found.

$$\mathbf{r}(t) = \left\langle t^2, 1 - t, \sqrt{t} \right\rangle \tag{9}$$

$$\mathbf{r}(t) = \left\langle 1, -1, e^{4t} \right\rangle \tag{11}$$

$$\mathbf{r}(t) = \langle t, 2\sin t, 3\cos t\rangle (t = \pi/6) \tag{19}$$