Arc length. If the vector function $\mathbf{r}(t)$ is thought of as giving position as a function of time, then its derivative $\mathbf{r}'(t)$ gives **velocity**. The length of $\mathbf{r}'(t)$ measures the **speed** and we shall see that the distance traveled along the curve is the integral of speed. We have already met **T**, which is a unit vector in the direction of $\mathbf{r}'(t)$. If a different parameter u is used to describe the curve, with t being an increasing function of u, $d\mathbf{r}/du = (dt/du)(d\mathbf{r}/dt)$. The first factor is a scalar, so it does not affect T.

The usual approach to measuring the length of a curve $\mathbf{r}(t)$ between the point where $t = a$ and the point where $t = b$ is to select values $a = t_0 < t_1 < t_2 < \ldots < t_n = b$ and find the length of the polygonal path connecting the points $\mathbf{r}(t_i)$ in order. This gives a sum of terms of the form

$$\Delta t\sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2}.$$

With any kind of luck, this will approach

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

as the t_i get closer together. We do not prove this formula, but we accept it as sufficiently plausible to be taken as a definition of arc length.

One encouraging fact is that, when the parameterization is changed by the substitution $t = g(u)$ with a monotonic function g, the integral doesn't change. That is, our way of finding the length of the curve depends on the curve and not on how it is drawn.

Exercises. There are only a few special curves for which this integral can be evaluated in closed form. Note how the examples here simplify.

$$x = 2\sin t \quad y = 5t \quad z = s\cos t \tag{1}$$

$$x = \sqrt{2}t \quad y = e^t \quad z = e^{-t} \tag{3}$$

Parameterization by arc length. As long as we know that a function is defined, whether or not we have previously named it, it is available for use. The arc length integral always gives arc length s as a function of the original parameter t. The derivative ds/dt is the integrand of the arc length integral, which is always positive. Thus, s is an increasing function of t, and there is an inverse function giving t in terms of s. This parameterization is often used to give a geometric definition of a quantity that we intend to study. In order to compute such quantities, a substitution is made to express it in terms of the parameter t appearing in the original definition of the curve. In particular, **T** is the derivative of the position vector with respect to arc length.

A useful result. The usual rules of calculus for sums and products are easily proved for derivatives of vectors. One consequence of this is that if $\mathbf{a}(t)$ is of constant length, so that $\mathbf{a}(t) \cdot \mathbf{a}(t)$ is a constant function, then

$$0 = \mathbf{a}(t) \cdot \mathbf{a}'(t) + \mathbf{a}'(t) \cdot \mathbf{a}(t) = 2\mathbf{a}(t) \cdot \mathbf{a}'(t)$$

so that $\mathbf{a}'(t)$ is always perpendicular to $\mathbf{a}(t)$.

Normals and curvature. From the last result, we get that $\mathbf{T}'(t)$ is perpendicular to $\mathbf{T}(t)$. The direction of $\mathbf{T}'(t)$ is called the **principal normal** and denoted **N**. Changing the parameter multiplies the derivative of **T** by a scalar (positive if the parameters are increasing functions of one another), so **N** is independent of the parameterization. If you take *arc length* as the parameter, then the magnitude of the derivative is also significant. This value is called **curvature**, and denoted κ, here described by **definition (8)**. Finally, it is not actually necessary to construct this parameterization, since the value at any point can be found from the chain rule. This gives **formula (9)**, which we use in the example below. However, this gives all geometric features as functions of the original parameter.

The main example. If $\mathbf{r}(t) = \langle a\cos t, a\sin t\rangle$, describing a circle of radius a in the xy plane, $\mathbf{r}'(t) = \langle -a\sin t, a\cos t\rangle$, and we can *see* its length and direction: $ds/dt = a$ and $\mathbf{T} = \langle -\sin t, \cos t\rangle$. Then

$$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{ds/dt} = \frac{\langle -\cos t, -\sin t\rangle}{a}$$

in this case. Geometrically, we see that **N** is a unit vector pointing towards the center of the circle, and $\kappa = 1/a$.

This portion of previously prepared slides will appear later, when the second formula for curvature is discussed.

Exercises. Some exercises for finding **T**, **N** and κ are

$$x = \tfrac{1}{3}t^3 \quad y = t^2 \quad z = 2t \tag{13}$$

$$x = \sin t \quad y = \cos t \quad z = \sin t \tag{17}$$

Domain and range. Before getting into the calculus, there are some theoretical issues. The **domain** of a function $f(x, y)$ is the set of pairs $(x, y) \in \mathbb{R}^2$ for which you are allowed to use the formula. It should take into account rules like *do not divide by zero*, or *only take square roots of nonnegative numbers*, but some *arbitrary* restrictions may be added to the description of the function. It is almost always easy to find the domain of a function.

The **range** is the set of numerical values that can be attained by the function. For simple examples, you are able to find the range “by inspection”, but the general problem of finding the range includes finding the maximum and minimum of the function, which is the goal of the chapter.

Exercises 14.1

7 For $f(x, y) = e^{x^2-y}$, find the domain and range of f and the value of $f(2, 4)$.

9 For $f(x, y, z) = x^2 \ln(x - y + z)$, find the domain and range of f and the value of $f(3, 6, 4)$.

Motivation from geometry and calculation. The tangent line to the graph of a function was central to many of the applications of single variable calculus. One way to express th property of the tangent is Taylor’s formula

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(\xi)}{2}(x - x_0)^2.$$

If x_0 is a number, an equation saying that y equals the sum of the first two terms on the right is the equation of a line. The last term is an **error term** giving the difference between the value $f(x)$ on the given curve and the y coordinate of the point on the tangent line for the same x. The ξ in this formula is a value between x_0 and x whose *existence* is asserted by Taylor’s theorem although no attempt is made to find it. Instead, one uses its rough location to argue that $\left|f''(\xi)\right|$ is not too large. When $|x - x_0|$ is small, this error term is not just the smallest term in the expression, but **much** smaller than the other terms. This says that the function may be reasonably well approximated by the tangent line in some interval around x_0. The tangent lines of space curves met in Chapter 13 have

similar properties although proofs look a little different because space curves, including lines in space are defined parametrically.