

Motion in Space. These formulas are used in mechanics. If $\mathbf{r}(t)$ represents the **position** of a body as a function of time, then $\mathbf{r}'(t)$ is **velocity** and $\mathbf{r}''(t)$ is **acceleration**. The key idea in Newton's explanation of motion was that motion represented the effect of **forces** and **force** is **mass** times **acceleration**. All of these quantities except **mass** are vectors; **mass** is a scalar that is constant for ordinary objects. If you observe the position function $\mathbf{r}(t)$, you determine the acceleration $\mathbf{r}''(t)$ and use that to help identify the force.

Exercises. No claim of physical significance for these exercises will be made. They serve only to illustrate the Calculus using the language of mechanics. The general instructions are to use given quantities to find all of **position** $\mathbf{r}(t)$, **velocity** $\mathbf{v}(t) = \mathbf{r}'(t)$, **speed** $v = ds/dt$, and **acceleration** $\mathbf{a}(t) = \mathbf{r}''(t)$ using the given information.

$$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle \quad (9)$$

$$\mathbf{r}(t) = \langle \sqrt{2}t, e^t, e^{-t} \rangle \quad (11)$$

$$\mathbf{a}(t) = \mathbf{k} \quad \mathbf{v}(0) = \mathbf{i} - \mathbf{j} \quad \mathbf{r}(0) = \mathbf{0} \quad (15)$$

A more elaborate problem is #19, which gives the motion

$$\mathbf{r}(t) = \langle t^2, 5t, t^2 - 16t \rangle$$

and asks for the time at which the speed is largest.

Components of acceleration. The formula

$$\mathbf{r}''(t) = \frac{d^2s}{dt^2} \mathbf{T} + \frac{ds}{dt} \mathbf{T}'$$

was derived in Section 13.3 (page 852). We also have $\mathbf{T}' = \kappa(ds/dt)\mathbf{N}$, so

$$\mathbf{r}''(t) = \frac{d^2s}{dt^2} \mathbf{T} + \kappa \left(\frac{ds}{dt} \right)^2 \mathbf{N}.$$

Writing v in place of ds/dt gives formula (7) of Section 13.4. The quantity v represent the **speed** of the object. The vectors \mathbf{T} and \mathbf{N} are perpendicular unit vectors that are part of a coordinate system that moves with the object. In particular \mathbf{T} is "straight ahead". In this coordinate system, the first part of the expression

for $\mathbf{r}''(t)$ describes the part of the acceleration (and, hence, of the force) that leads to a change of speed, while the second part describes the part of the acceleration that leads to a change of direction. These two terms are important in the way that motion is perceived, so it is important to see how they can be computed. Since there are many different approaches to finding the quantities in this formula, it is useful to point out that calculations done earlier with numerical vectors give a way to organize the work efficiently. One needs only connect the use of the word *component* here with the earlier use of that word.

The velocity vector $\mathbf{v}(t) = v(t)\mathbf{T}(t)$, so it defines the direction $\mathbf{T}(t)$. In the previous sense of the word, the component of $\mathbf{a}(t)$ in the direction of $\mathbf{v}(t)$ is what we call here “the tangential component of acceleration”, so it will be equal to d^2s/dt^2 even if its computation does not appear to involve differentiation of $v(t) = ds/dt$. If you have this component, you also have the projection by multiplying by the vector $\mathbf{T}(t)$. From the whole vector and the tangential projection, you can find the projection on the principal normal and its

component. This leads to an *algorithm* for computing curvature that is not easily summarized in a formula, but may be simpler than the formulas of section 13.3.

Exercises. The instructions are to find the tangential and normal components of acceleration.

$$\mathbf{r}(t) = \langle 3t - t^3, 3t^2 \rangle \quad (29)$$

$$\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle \quad (31)$$

Definition of limit. It is useful at this point to review the definition of limit, extended to cover functions of several variables. For simplicity, we use only two variables.

We say that

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L \quad (*)$$

if we can force $f(x, y)$ to be arbitrarily close to L (within an ϵ given by *el Exigente*) by demanding that (x, y) be sufficiently close to (a, b) (at a distance no more than a δ that we choose), independent of direction.

Note that it is the whole sentence $(*)$ that is defined. The use of the equal sign in that definition is justified by

The main theorem. Limits are unique. That is, given the function f and the point (a, b) , there is *at most one* choice of L satisfying $(*)$.

Proof. If not, let L_0 and L_1 be two different numbers that both satisfy $(*)$. The triangle inequality gives

$$|L_0 - L_1| \leq |L_0 - f(x, y)| + |f(x, y) - L_1| \quad (\dagger)$$

for any point (x, y) in the domain of f . If you persist in claiming that L_0 and L_1 both satisfy $(*)$, I calculate

$$d = |L_0 - L_1|.$$

To say that $L_0 \neq L_1$ is to say that $d > 0$ (strictly). Then I choose $\epsilon = d/2$. Then, if (x, y) is any point that you think is close enough to (a, b) , then both terms on the right side of (\dagger) are less than $d/2$, which contradicts (\dagger) .

Non-existence of limits. The idea of this proof is used to identify cases in which limits do not exist. If there are two different values v_0 and v_1 such that $f(x, y) = v_0$ for some points (x, y) arbitrarily close to (a, b) while $f(x, y) = v_1$ for other points arbitrarily close to (a, b) , then it is not possible for $f(x, y)$ at *all* points (x, y) close to (a, b) to be within $|v_0 - v_1|/2$ of any single value.

Continuity. Almost every function that we know how to express has the property that, at points where the expression for the function can be evaluated, that value may be used for L in (*). This property is called **continuity**. Continuity means that that the value of a function at a point can be approximated by evaluating the function at a nearby point. This is what we do every time we trust our calculators to do much more than verify that $1 + 1 = 2$, so continuity is an abstraction of the idea of *computable*. The rules for building functions that we use are easily seen to preserve continuity, although complicated functions (for which δ is small) may require extraordinary care to give reasonable accuracy.

Exercises 14.2

Investigate limits of the following expressions at any point at which the expressions is not obviously continuous.

$$\frac{x^2}{x^2 + y^2} \quad (7)$$

$$\frac{xy}{\sqrt{x^2 + y^2}} \quad (11)$$

$$e^{-xy} \sin \pi z/2 \quad (17)$$

$$\frac{x^2 y^3}{2x^2 + y^2} \quad (35)$$