

The cross product. The formula summarized by

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad (*)$$

clearly gives a vector perpendicular to

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle \text{ and } \mathbf{b} = \langle b_1, b_2, b_3 \rangle,$$

using a systematic formula. It is zero when $\mathbf{a} \parallel \mathbf{b}$, and only in this case. Direct computation (Theorem 6) shows that the length of this vector is the product of the lengths of \mathbf{a} and \mathbf{b} and the sine of the angle between them. This shows that the length is the area of the parallelogram spanned by \mathbf{a} and \mathbf{b} . The only surprise is that interchanging the roles of \mathbf{a} and \mathbf{b} sends the product into its negative. Formula (*) defines the cross product $\mathbf{a} \times \mathbf{b}$. In addition to giving directions perpendicular to two given directions and finding areas, it gives a computational way to describe **orientation** in space — the distinction between *left* and *right*, because it is **skew commutative**.

This product is also **linear** in each factor. There are also rules for interpreting triple product — items 5 and 6 of theorem 8 that we won't do much with.

Exercises 12.4

Compute $\mathbf{a} \times \mathbf{b}$.

$$\mathbf{a} = \langle 1, 2, 0 \rangle \quad \mathbf{b} = \langle 0, 3, 1 \rangle \quad (1)$$

$$\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} \quad \mathbf{b} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k} \quad (7)$$

Method: Use determinant description; or expand by linearity and employ the multiplication table for \mathbf{i} , \mathbf{j} and \mathbf{k} .

#25. Find vector orthogonal to plane through

$$P(1, 0, 0), \quad Q(0, 2, 0), \quad R(0, 0, 3)$$

and the area of $\triangle PQR$.

Method Introduce vectors \overrightarrow{PQ} and \overrightarrow{PR} , and relate desired quantities to properties of cross product.

The equation of a plane. A single equation

$$ax + by + cz = d \quad (*)$$

defines a plane. To see this, suppose you have one solution (x_0, y_0, z_0) of $(*)$, which could be obtained by setting $x = x_0$ and $y = y_0$ arbitrarily and solving $(*)$ for z to find z_0 . Then $(*)$ is equivalent to

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \quad (**)$$

This says that the general vector in the plane $(**)$ $\overrightarrow{P_0P}$ is perpendicular to $\mathbf{n} = \langle a, b, c \rangle$. This leads to a geometric way of recognizing the solutions of $(*)$ as a plane. It also says that the equation can be written if one knows a point in the plane and a direction perpendicular to the plane.

The plane through three points. A common way to describe a plane is to give three points P_0, P_1, P_2 in the plane. Then P_0P_1 and P_0P_2 are two directions in the plane and $P_0P_1 \times P_0P_2$ is perpendicular to the plane. This, together with the point P_0 in the plane gives the information needed to write the equation.

Intersection of line and plane. In general, an equation gives a condition on the coordinates (x, y, z) of a general point for it to lie in a set. The intersection of a line, given in parametric form, and a plane, given by an equation, is found by substituting the parametric description of the points on the line into the equation for the plane. This gives a single equation in the parameter. Solve this equation and use that value in the description of the line to find the coordinates of the intersection.

Intersection of two planes. The easiest way to find the line of intersection of two planes is to use the geometric interpretation of the equation of a plane. The coefficients give a direction perpendicular to the plane, i.e. perpendicular to each direction in the plane. The direction of the line of intersection is thus perpendicular to the two vectors giving the directions of the planes, so its direction is given by the cross product. It remains to find a point on the line. This can be done by choosing the z -coordinate arbitrarily ($z = 0$ is a good choice). The equations of the planes then give two equations in x and y that are easy to solve simultaneously.

Exercises

#23. Find the equation of the plane through the origin and parallel to $2x - y + 3z = 1$.

Method. Reformat given numbers.

#27. Find the equation of the plane through $(0, 1, 1)$, $(1, 0, 1)$, and $(1, 1, 0)$.

Method. Find directions in the plane, then perpendicular one to the plane.

#31. Find the equation of the plane through $(6, 0, -2)$ that contains the line $x = 4 - 2t$, $y = 3 + 5t$, $z = 7 + 4t$.

Method. Begin by finding directions in the plane.

#35. Find the point where the the line $x = 1 + t$, $y = 2t$, $z = 3t$ meets the plane $x + y + z = 1$.

Method. Find the value of t that locates the point on the line.

Motivation from geometry and calculation. The tangent line to the graph of a function was central to many of the applications of single variable calculus. One way to express the property of the tangent is Taylor's formula

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(\xi)}{2}(x - x_0)^2.$$

If x_0 is a number, an equation saying that y equals the sum of the first two terms on the right is the equation of a line. The last term is an **error term** giving the difference between the value $f(x)$ on the given curve and the y coordinate of the point on the tangent line for the same x . The ξ in this formula is a value between x_0 and x whose *existence* is asserted by Taylor's theorem although no attempt is made to find it. Instead, one uses its rough location to argue that $|f''(\xi)|$ is not too large. When $|x - x_0|$ is small, this error term is not just the smallest term in the expression, but **much** smaller than the other terms. This says that the function may be reasonably well approximated by the tangent line in some interval around x_0 . The tangent lines of space curves met in Chapter 13 have

similar properties although proofs look a little different because space curves, including lines in space are defined parametrically.

The definition of the tangent plane will require that one plane approximates the surface near a point uniformly in all directions. Some surfaces that are otherwise well-behaved, like the cone $z^2 = x^2 + y^2$ fail to have such a tangent plane at the origin. Although it has many planes with some of the properties of a tangent, it is better not to try to weaken the definition to allow more tangent planes. The strict requirement has so many useful consequences, and is satisfied in many cases, that little is lost by leaving a few examples out of the theory.

Derivatives and tangent planes. The geometric version of the existence of a derivative of a function f at a point (a, b) is the existence of a tangent plane to the surface $z = f(x, y)$ at the point where $x = a$ and $y = b$. A tangent plane has an equation of the form $z = Ax + By + C$ for constants A, B and C , which we abbreviate $z = L(x, y)$ — L standing for **linear**. The definition giving the most efficient characterization of

tangent planes is to require that, for all $\epsilon > 0$,

$$|f(x, y) - L(x, y)| < \epsilon \sqrt{(x - a)^2 + (y - b)^2}$$

for all (x, y) sufficiently close to (a, b) , independent of direction. The expression on the right is chosen so that given any nonzero linear expression L , it must fail to bound L close to (a, b) for some ϵ . This allows the proof of the main theorem to be modified to show that there is at most one such L .

Using tangent lines to find tangent planes. Tangent lines to curves could have been defined in the same way. Hence, intersecting with the plane $y = b$ gives a curve whose equation is $z = f(x, b)$ and a tangent line to this curve at the point where $x = a$. Since tangent lines could have been characterized by the same ϵ - δ definition as tangent planes, these tangent lines *must* lie in the tangent plane. Finally, writing the equation of the tangent plane in the form

$$z = L(x, y) = A(x - a) + B(y - b) + C$$

we can use what we know about tangent lines to show that $C = f(a, b)$, A is the derivative of $f(x, b)$ with

respect to x evaluated at $x = a$, and B is the derivative of $f(a, y)$ with respect to y evaluated at $y = b$.

Exercises 14.4 Find equation of tangent plane at indicated point.

$$z = y^2 - x^2 \quad (-4, 5, 9) \quad (1)$$

$$z = \sqrt{4 - x^2 - 2y^2} \quad (1, -1, 1) \quad (3)$$

$$z = \ln(2x + y) \quad (-1, 3, 0) \quad (5)$$

$$z = e^x \cos(xy) \quad (0, 0, 1) \quad (13)$$