Maxima and minima. Everyone's favorite problem of differential calculus is the determination of maxima and minima of a function on a region.

An important theoretical result asserts that, if the function is **continuous** and the region **closed** and **bounded** in some $\mathbb{R}^n$, then these extreme values exist and are attained at points of the domain.

Calculus improves this by saying that, if the function is **differentiable**, then the points at which the extreme values are taken on are **easy to find**. Note that this shifts the emphasis from the values of the function to the points in the domain where the function takes those values.

To apply the methods of one-variable calculus to functions of several variables, we need the following

Secret Weapon. *If $A \subseteq B$ and if the maximum of a function f on B is taken on at a point $x \in A$, then $f(x)$ is also the maximum of f on A.*

Proof. Think about it!

In applications, B will be the given region in $\mathbb{R}^n$ and A will be a curve lying in B. Composing the given function on B with the parameterization of A gives a real valued function of a real variable, and one-dimensional calculus applies.

You may protest that we don't know A, but that will be dealt with.

What one-variable calculus gives us is a **necessary condition** for a point to be a maximum or a minimum. If we can find **any** curve A through a point P for which this condition fails at P, then $f(P)$ is not an extreme value of f on B and we can look somewhere else.

It turns out that points on the boundary of B behave differently than interior points. Interior points are easier, so we do them first.

The key special case. What then do we take as A? The simplest examples turn out to suffice: let A be a line segment parallel to one of the coordinate axes and lying in B. Every interior point has segments through it parallel to each coordinate axis. The coordinate on the axis can be taken as the parameter on the segment.

The derivative with respect to this parameter is exactly the partial derivative of the given function with respect to the selected coordinate. The **secret weapon** then implies

Theorem. *If f takes on its maximum (or minimum) value on the set B at an interior point P, then all partial derivatives of f are zero at P.*

Since the partial derivatives give all components of the gradient, such points P have $\nabla f(P) = 0$.

Interior points that are not ruled out by this result are called **critical points** of the function.

The second derivative test. In single-variable calculus, most critical points are **local minima** or **local maxima**. A sufficient condition for a local minimum is positive second derivative, and for a maximum is a negative second derivative.

For functions of several variables, taking our secret weapon into account, we have that a minimum point cannot have a negative second derivative in any direction and a maximum point cannot have a positive

second derivative in any direction, so critical points at which second derivatives of both signs can be found are neither maxima nor minima. Such points are called **saddle points**.

On the line $x = x_0 + at, y = y_0 + bt$, the first derivative of $f(x, y)$ is $f_1(x, y)a + f_2(x, y)b$, so the second derivative is $f_{11}(x, y)a^2 + f_{12}(x, y)ab + f_{21}(x, y)ba + f_{22}(x, y)b^2$. There is a fully developed theory of such expressions, but we will only extract a simple criterion for recognizing the types of critical points for functions of two variables.

First, notice that if all second derivatives are positive at a point, then f_{11} must be positive at that point. That is, if the second derivative has a constant sign, it is the sign of f_{11}. Completing the square in our expression for the second derivative gives

$$\frac{1}{f_{11}}\Big((f_{11}a + f_{12}b)^2 + (f_{11}f_{22} - f_{12}^2)b^2\Big).$$

Thus, all second derivatives have the same sign if $f_{11}f_{22} - f_{12}^2 > 0$ and there are directions with second derivatives of opposite sign if this expression is negative.

Exercises 14.7. Most of the exercises in Section 14.7 ask to find critical points and classify them using the second derivative test. There is also a suggestion that graphs be constructed to aid in discovering features. A Maple worksheet with solutions of these problems has been prepared.

#3. $f(x, y) = 4 + x^3 + y^3 - 3xy$.

#7. $f(x, y) = x^2 + y^2 + x^2y + 4$.

#19. $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$.

Global max and min. If there are only a few critical points, it will be easier to list them all with the value of the function at each, than to perform a second derivative test. The chief interest in the test is theoretical. In particular, it demonstrates the existence of critical points that can be neither minima nor maxima because there are both positive and negative second derivatives. A simple example is $x^2 - y^2$. Unfortunately, most calculus textbooks inflate the importance

of the second derivative test, which only determines the *local* behavior of the function, while downplaying the idea that there is usually a short list of points that are possible locations of the extreme values of the function. We will give an example of a complete analysis of a function on a closed and bounded set at the end of the lecture.

Lagrange multipliers. We now characterize critical points on the boundary of a region. Let P be a point on the boundary of the set B.

Suppose first that a smooth curve can be drawn on the boundary of B *through* P. That is, there is a vector function $\mathbf{r}(t)$ whose domain contains a neighborhood of 0 and whose range lies entirely in the boundary of B, with $\mathbf{r}(0) = P$, and $\mathbf{r}'(0) \neq \mathbf{0}$. The last part is the *smoothness* condition. (If B is a polygon, then any $\mathbf{r}(t)$ for which $\mathbf{r}(0)$ is a vertex of the polygon must have $\mathbf{r}'(0) = \mathbf{0}$ if the derivative exists. This follows from the fact that a nonzero $\mathbf{r}'$ must have a well defined direction, but this direction changes when when one goes through a vertex.) Then the restriction of f to B can be represented as $f(\mathbf{r}(t))$, and the derivative with respect to t is $\nabla f \cdot \mathbf{r}'(t)$. At any max or min on this curve, we must then have that $\mathbf{r}'(t)$ is perpendicular to ∇f.

Now, suppose that B is given in the form

$$\left\{ \mathbf{x} \in \mathbb{R}^n : g(\mathbf{x}) \leq 0 \right\}.$$

The boundary is given by $g(\mathbf{x}) = 0$. Hence, the composition $g(\mathbf{r}(t))$ is a constant function, so must have derivative zero. The chain rule, evaluated at $t = 0$, gives

$$\nabla g(P) \cdot \mathbf{r}'(0) = 0.$$

This says that $\mathbf{r}'(0)$ must be perpendicular to $\nabla g(P)$. If the boundary of B is smooth at P, then curves $\mathbf{r}(t)$ can be found for which $\mathbf{r}'(0)$ takes all values allowed by this condition. However, we have already seen that

$$\nabla f(P) \cdot \mathbf{r}'(0) = 0.$$

In words, every vector perpendicular to $\nabla g(P)$ must also be perpendicular to $\nabla f(P)$. For points at which the boundary is smooth, this forces the existence of

a value traditionally called λ for which $\nabla f(P) = \lambda \nabla g(P)$. In $\mathbb{R}^n$ this gives n equations in λ and the n coordinate functions. The condition, $g(P) = 0$ gives one more equation. We not have as many equations as we have variables, and we should be able to solve this system of equations. Unfortunately, the algebra for doing this is sometimes difficult. A systematic approach to this algebra is possible, but we don't have time to develop one. One practical consequence of this is that only simple examples will be considered.

This approach can be extended to deal with smooth points of an object of any dimension d in any $\mathbb{R}^n$, but few examples reduce to algebra that can be done easily.

There is one easy application of Lagrange multipliers that appears in many variants. There are two forms that reduce to the same algebra:

(1) Let x and y be nonnegative with $x + y = s$ (s constant). Find the maximum of xy.

(2) Let x and y be positive with $xy = p$ (p constant). Find the minimum of $x + y$.

Since our notation for gradients uses functions rather than expressions, introduce the functions $S(x, y) = x + y$ and $P(x, y) = xy$. Then the Lagrange multiplier method for either problem (1) or problem (2) says that extreme values occur when $\nabla S \parallel \nabla P$. Since $\nabla S = \langle 1, 1\rangle$ and $\nabla P = \langle y, x\rangle$, this says $x = y$. In (1), the given constraint then gives $x = y = s/2$ and $xy = s^2/4$. In (2), we get $x = y = \sqrt{p}$ and $x + y = 2\sqrt{p}$.

In each problem, the method selects a unique point. The only other candidates for the location of extreme values of the function are the endpoints. In (1), the product is zero at both endpoints; in (2), there are no true endpoints, but $x + y \to \infty$ outside bounded parts of the curve. This explains how we knew that the interior point gave a maximum in (1) and a minimum in (2).

Exercises 14.8.

#5. $f(x, y) = x^2 y$ on $x^2 + 2y^2 = 6$.

#17. $f(x, y, z) = yz + xy$ on intersection of $xy = 1$ and $y^2 + z^2 = 1$.

Global max-min exercises. Here we are to find the absolute maximum and minimum of $f(x, y)$ on a closed, bounded set D. The set will consist of: (1) **interior points**, for which we need to find where $\nabla f = 0$; (2) **smooth boundary arcs**, for which we need to find where ∇f is perpendicular to the arc; and (3) **corners**, which are always considered.

14.7#27. $f(x, y) = 5 - 3x + 4y$ on triangle with vertices (0, 0), (4, 0), (4, 5).

14.7#31. $f(x, y) = 1 + xy - x - y$ on $x^2 \le y \le 4$.