Applications of double integrals. An easy way to claim that an integral arises as an application is to say that the integrand represents a **density**. This is mostly a matter of definition: if matter is spread unevenly over a plane region, it is reasonable to find an *average amount* over a piece of the region by cutting out the piece, weighing it, and dividing the wright by the area of the piece. If the piece is small enough, this will be close to a quantity that is *weight per unit area at a point*. It is more accurate physically to treat the weight as an indirect measurement of *mass*. The same approach could be applied to other forms of physical *content*, like *electrical charge*, but I will use mass for most of the examples. It is a form of the fundamental theorem of calculus that the total content of the region is the integral of density. Thus, if there is some way in which the density is known, integration recovers to total content of a region.

Unfortunately, the density that is most likely to be known in advance is a *uniform* density. Then, mass is simply the product of area and density. The methods of single-variable calculus were often powerful

enough to compute areas with a plausible argument that the integral was computing the intended area. Double integrals are much easier to justify, since they are based on a more refined type of Riemann sum, and they allow related computations to be formulated using a unified argument. In particular, the **moment** of a planar object with respect to a line is is found by integrating the *signed distance* to that line with respect to mass. In single-variable calculus, this was done when the line was one of the coordinate axes, but any such integral can be easily formulated and evaluated as a double integral. The textbook is a little timid about approaching this topic and continues to emphasize moments with respect to the coordinate axes. There are also **second moments**, for which the integrand is the *square* of the distance to a line (or a point). The physical term for a second moment is *moment of inertia*. One benefit of having simple expressions for these quantities is that one can prove properties of the moments by considering properties of the quantity that is integrated to obtain that moment.

Theorems about moments. Consider a region $\mathcal{R}$ represent a physical object in the plane whose total mass is M.

1. If L_1 is the line at signed distance d from line L_0, and the moment of $\mathcal{R}$ about L_i is denoted μ_i, then

$$\mu_0 = \mu_1 + Md$$

1a. Given L_0, one can find a parallel line L_1 such that $\mu_1 = 0$.

2. If the moments of $\mathcal{R}$ about two different lines through a point P are zero, then the moment of $\mathcal{R}$ about any line through P is zero. The point P is called the **center of mass** of $\mathcal{R}$.

3. The second moment of $\mathcal{R}$ about a line L is always nonnegative, so it can be denoted σ^2.

3a. If L_0 passes through the center of mass of $\mathcal{R}$ and L_1 is parallel to L_0 at distance d, and if the second moment of $\mathcal{R}$ about L_i is σ_i^2, then

$$\sigma_1^2 = \sigma_0^2 + d^2.$$

4. The second moments about three different lines through the center of mass determine the second moments about all lines. There is an ellipse having the same second moments as $\mathcal{R}$.

A sufficiently general calculation of the center of mass of a figure of a particular shape gives a result which has a geometric interpretation. A few of these are worth remembering. For the immediate needs of exams in this course, knowing the answer will mainly be useful as a check, since exam questions are likely to emphasize setting up the integral and showing some details of its calculation. However, once people accept that you have learned the calculus, there is little reason to do the these calculations. Indeed, a poor choice of coordinates can give you an integral that is not easy to evaluate even though it has a simple value.

Example: Center of mass of a triangle.

Another advantage of reducing the calculation to an integral is that the integral may be evaluated by transforming to polar coordinates or using a special parameterization of the boundary to simplify the line integral arising from interpreting the integral as in the proof of Green's theorem.

Example: Center of mass of a circluar sector.

Probability. Densities also appear in the study of probability. A single measurement typically has a **probability density function** on the ilne. The special features of such functions are thatit is everywhere nonnegative and its integral over the whole line is 1. There was an introduction to this subject in Section 8.5. With two measurements, a density function of oredered pairs of measuremnts can be defined. This is a nonnegative function on $\mathbb{R}^2$ whose integral over the whole plane is 1. The first moments with respect to the axes are called **averages** and second moments with respect to a line through the center of mass are called **variance**.