Areas in a plane. Suppose we have vectors

$$\mathbf{v}_0 = \langle a_0, b_0, c_0 \rangle \quad \mathbf{v}_1 = \langle a_1, b_1, c_1 \rangle$$

based at a point $P(x_0, y_0, z_0)$ in $\mathbb{R}^3$. Then the four points P, $P+\mathbf{v}_0$, $P+\mathbf{v}_1$, $P+\mathbf{v}_0+\mathbf{v}_1$ are the vertices of a parallelogram $\mathcal{P}$ in space, which lies in the plane

$$Ax + By + Cz = D$$

where

$$\langle A, B, C \rangle = \mathbf{v}_0 \times \mathbf{v}_1$$

and

$$D = Ax_0 + By_0 + Cz_0.$$

The area of $\mathcal{P}$ is

$$|\mathbf{v}_0 \times \mathbf{v}_1| = \sqrt{A^2 + B^2 + C^2}.$$

If we project this figure into the xy plane, we get a parallelogram with one vertex at $P_0(x_0, y_0, 0)$ and sides given by the vectors

$$\mathbf{w}_0 = \langle a_0, b_0, 0 \rangle \quad \mathbf{w}_1 = \langle a_1, b_1, 0 \rangle$$

whose area is

$$|\mathbf{w}_0 \times \mathbf{w}_1| = |\langle 0, 0, C \rangle| = |C|$$

The ratio of the area of the projection to the area of $\mathcal{P}$ is

$$\frac{|C|}{\sqrt{A^2 + B^2 + C^2}}.$$

Note that this quantity does not change if A, B, C, and D are all multiplied by a number λ to obtain a different equation of the same plane.

Also note that the projection has area zero if $C = 0$, which says that the equation of the plane does not depend on z, or that the plane is perpendicular to the xy plane.

On the other hand, if $C \neq 0$, we can invert this ration to find the amount that the area of the projection should be multiplied by to obtain the area of $\mathcal{P}$. This ratio has the form

$$\sqrt{\left(\frac{A}{C}\right)^2 + \left(\frac{B}{C}\right)^2 + 1},$$

so it is always at least 1. The quantities A/C and B/C in this formula are the negatives of the coefficients of x and y when the equation of the plane is solved for z.

Area on the graph of a function. The main principle in the differential calculus of functions of several variables is that, if you confine yourself to a set of small enough diameter, any reasonable function is approximately linear. If you really believe this, you are led to the conclusion that the area of of the part of the graph of $f(x, y)$ over a neighborhood of a point P_0 in the xy-plane can be approximated by considering the corresponding area in the tangent plane at the point of this surface above P_0. The sum of such areas for a partition of a region $\mathcal{D}$ in the xy-plane is a Riemann sum of

$$\iint_{\mathcal{D}} \sqrt{f_1(x, y)^2 + f_2(x, y)^2 + 1}\, dA. \qquad (A)$$

The quantity dA in this formula stands for $dx\,dy$ or $r\,dr\,d\theta$ in rectangular and polar coordinates, respectively.

A rigorous study of surface area is very difficult. Riemann integrals are defined in terms of a very general type of limit. The integrals exist under some fairly general assumptions, guaranteeing that the Riemann sums approximate the value of the integral found by calculus if the partition is fine enough. However, the Riemann sums just constructed involve approximating the surface by pieces that don't fit together to form an approximate surface. If we want to believe that this integral really does give surface area, it would be nice to connect it with the area of something that resembled the surface. A number of reasonable ideas for constructing such measurements turn out to be more general than Riemann sums and often fail to have limits. Although the values obtained from the integral (A) turn out to be correct whenever the integral makes sense, we cannot do a better job of relating them to geometric measurements of the surface.

There are only a few examples included in the exercises. The difficulty here is that most of the integrals obtained from (A) cannot be evaluated in terms of familiar functions. This difficulty was already present in connection with arc length. For example, the integral giving the perimeter of an ellipse usually cannot be expressed in terms of familiar functions. The use of (A) to set up an integral representing a surface area is one possible exercise. The warning, "Do not attempt to evaluate the integral", is given in such cases to signify that the result is not likely to be expressible in terms of familiar functions.

Surfaces of revolution. We show that this formula is consistent with the one used in section 10.3. For a surface of revolution given by a function, we have

$$z = f(r) \quad r^2 = x^2 + y^2.$$

Then $z_x = f'(r)r_x$ and $z_y = f'(r)r_y$ from the first equation, while $r_x = x/r$ and $r_y = y/r$ from the second. Thus, the area of the surface is given by integrating

$$\begin{aligned}\sqrt{1+z_x^2+z_y^2} &= \sqrt{1+\left(f'(r)\right)^2\left(\left(\frac{x}{r}\right)^2+\left(\frac{y}{r}\right)^2\right)}\\ &= \sqrt{1+\left(f'(r)\right)^2}\end{aligned}$$

with respect to area in xy-plane.

There is no special significance to the xy plane in these calculations. It is equally easy to use one of the other coordinate planes and the formulas will involve a different component of the vector perpendicular to the surface. Thus, if $\langle A, B, C\rangle$ is perpendicular to a surface at a point, then the element of surface area at that point dS satisfies

$$\frac{dS}{\sqrt{A^2+B^2+C^2}} = \left|\frac{dx\,dy}{C}\right| = \left|\frac{dx\,dz}{B}\right| = \left|\frac{dy\,dz}{A}\right|$$

Exercises 15.6

#3. Find the area of the portion of the plane $3x+2y+z=6$ that lies in the first octant.

#9. Find the area of the potion of $z = xy$ that lies inside $x^2+y^2=1$.