Parametric surfaces. Section 16.6 extends the study of surface area to parametric surfaces.

Just as curves in space are best described by giving a vector function $\mathbf{r}(t)$ that may be thought of as describing how the curve is drawn, so surfaces should be given by a function $\mathbf{r}(u, v)$ expressing the space coordinates x, y, and z in terms of two parameters u and v that play the role of coordinates on the surface.

Surfaces of revolution are naturally parameterized by the adding an angular parameter θ to the parameter that draws the curve being rotated. If you are rotating about one of the coordinate axes, the distance along that axis is part of the parametric description of the curve, and the other two coordinates are obtained by multiplying the distance to that axis (the other part of the parametric description) by $\cos\theta$ and $\sin\theta$.

Another family of surfaces that are easily obtained in *Maple* are the *tubeplots* the fatten up a space curve by identifying the points at a certain distance from the curve in the plane perpendicular to the curve at each point. One of the simplest examples if the **torus**. Start with the circle $r_0\langle\cos u, \sin u, 0\rangle$ and use the

vectors $\langle\cos u, \sin u, 0\rangle$ and $\langle 0, 0, 1\rangle$ as perpendicular unit vectors in the normal plane, so one gets $(r_0 + r_1 \cos v)\langle\cos u, \sin u, 0\rangle + r_1 \sin v\langle 0, 0, 1\rangle$.

You can also get the surface traced out by the tangent lines to a space curve by letting $\mathbf{r}(u)$ be the curve and v the parameter that draws the tangent line at $r(u)$.

In the direct approach to area of parameterized surfaces, we partition the region in the uv-plane into pieces of small diameter, and approximate the corresponding part of the surface by parallelograms in the tangent plane whose sides are the given by vectors $\mathbf{r}_1(u, v)\Delta u$ and $\mathbf{r}_2(u, v)\Delta v$, the area of the parallelogram is given by the length of the cross product of these vectors. As in our derivation of formula (A), this leads to

$$\iint |\mathbf{r}_1(u, v) \times \mathbf{r}_2(u, v)|\; du\, dv$$

over the region in the uv-plane parameterizing the part of the surface we are measuring. Comparing this to (A) in the case where x and y are functions of u and

v and $z = f(x, y)$, we find

$$\mathbf{r}_1 = \langle x_u, y_u, f_x x_u + f_y y_u\rangle$$
$$\mathbf{r}_2 = \langle x_v, y_v, f_x x_v + f_y y_v\rangle$$

so that $\mathbf{r}_1(u, v) \times \mathbf{r}_2(u, v)$ is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_u & y_u & f_x x_u + f_y y_u \\ x_v & y_v & f_x x_v + f_y y_v \end{vmatrix}$$

This leads to

$$du\, dv = \frac{dx\, dy}{|x_u y_v - x_v y_u|}.$$

The denominator is the third component of the cross product, and it is also the factor that will be found in section 15.9. In other words, this also corresponds to the formula that would be obtained by inventing the **implicit function** that gives $z = f(x, y)$ from the parameterization of the surface.

There is no special significance to the xy plane in these calculations. It is equally easy to use one of the

other coordinate planes and the formulas will involve a different component of the vector perpendicular to the surface. Thus, if $\langle A, B, C\rangle$ is perpendicular to a surface at a point, then the element of surface area at that point dS satisfies

$$\frac{dS}{\sqrt{A^2 + B^2 + C^2}} = \left|\frac{dx\, dy}{C}\right| = \left|\frac{dx\, dz}{B}\right| = \left|\frac{dy\, dz}{A}\right|$$

Exercises 16.6

#39 Find the area of the part of the sphere $x^2 + y^2 + z^2 = a^2$ the lies inside the cylinder $x^2 + y^2 = ax$.