Proof of Stokes' Theorem. It is conventional to state theorems before proving them, but this sometimes leads to unmotivated work aimed at establishing the definitions needed to state the result. Since the motivation lies in the proof, it might be better to give the proof first, and then interpret it. We use (x, y, z) for the coordinates in the $\mathbb{R}^3$ where all objects are constructed.

Our first object will be a surface $\mathcal{S}$ that we begin by assuming to be the graph of a function $z = g(x, y)$. This suffices for our needs since we need only have enough of a proof to provide clues to the correct statement of the theorem and interpretations of the formulas it relates.

The second object is a closed curve $\mathcal{C}$ lying in the surface $\mathcal{S}$. One side of the Stokes Theorem equation is the integral of a vector field $\mathbf{F}$ around $\mathcal{C}$. Such integrals usually require that $\mathcal{C}$ be given by a parameterization

$$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$$

in order to be calculated, so we reserve this notation. Since $\mathcal{C}$ is contained in $\mathcal{S}$, we must have $z(t) =$

$g(x(t), y(t))$. Let $\mathcal{C}_0$ be the projection of $\mathcal{C}$ into the xy-plane, so that it is parameterized by $\langle x(t), y(t)\rangle$.

The vector field

$$\mathbf{F} = \big\langle P(x, y, z),\ Q(x, y, z),\ R(x, y, z)\big\rangle$$

Then $\oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$ is an abbreviation for the expression

$$\int_a^b P(x(t), y(t), z(t))x'(t)+ \\ Q(x(t), y(t), z(t))y'(t)+ \\ R(x(t), y(t), z(t))z'(t)\,dt,$$

where the values $t = a$ and $t = b$ correspond to going once around $\mathcal{C}$. This integral is a sum of three terms that can be treated separately.

We next write expressions for the interpretations of these integrals as line integrals on $\mathcal{C}_0$. This involves replacing every mention of z by $g(x, y)$. This is straightforward in the first two terms, but in the third term,

$$z'(t) = g_x(x(t), y(t))x'(t) + g_y(x(t), y(t))y'(t).$$

Abbreviating everything to get the correct expression in terms of t gives

$$\oint_{\mathcal{C}_0} P(x, y, g(x, y))+ \\ R(x, y, g(x, y))g_x(x, y)\,dx+ \\ Q(x, y, g(x, y))+ \\ R(x, y, g(x, y))g_y(x, y)\,dy.$$

This is a line integral in the xy plane, so it is equal to the double integral over the region $\mathcal{S}_0$ bounded by $\mathcal{C}_0$ of

$$\frac{\partial}{\partial x}\big(Q(x, y, g(x, y)) + R(x, y, g(x, y))g_y(x, y)\big)- \\ \frac{\partial}{\partial y}(P(x, y, g(x, y)) + R(x, y, g(x, y))g_x(x, y)).$$

The terms containing second derivatives of g or products of two derivatives of g in this expression cancel, and the remaining terms may be grouped as

$$(R_y - Q_z)(-g_x) + (P_z - R_x)(-g_y) + (Q_x - P_y).$$

The second factors in these terms are the components of $\big\langle -g_x, -g_y, 1\big\rangle$ which is perpendicular to $\mathcal{S}$. The first factor must then be the components of a vector field being integrated over the surface. Note that the normalization of our normal vector to have third coordinate 1 gives the expression for a flux integral for the upward orientation with respect to $dx\,dy$.

The other factor is a vector field constructed from derivatives of $\mathbf{F}$. This expression is often described as $\nabla\times\mathbf{F}$ since the pattern of partial derivatives follows the same pattern as terms in a cross product. This construction is called the **curl** of $\mathbf{F}$.

From particular to general. The previous analysis applies on any piece of the surface where z **can be given** as a function of x and y. A computation of the surface integral using parameters other than x and y only requires a different *scaling* of the normal vector. That is, we consider a flux integral of a vector field to be the integral of the dot product of the vector field with $d\mathbf{S}$ which is a product of a suitably scaled normal vector with an element of area in the parameter space. Whenever two parameterizations are possible, the ratio of the area elements can be combined with the normal vector to give the relation between the scaled normals.

In particular, the rightward orientation for an integral with respect to y and z would have first coordinate $+1$.

All of our oriented integrals are such that cutting the region into pieces gives the integral as the sum over the pieces. The implicit function theorem tells us that a nonzero component of the normal at a point allows a small piece containing the point to be found on which the selected variable is a function of the remaining variables. The whole surface is thus broken into pieces covered by our special case.

Conservative vector fields. One observation connected to Stokes' Theorem is that the components of $\nabla \times \mathbf{F}$ are exactly the things that must be tested to show that $\mathbf{F}$ is conservative. Since the line integral is zero around every curve, it is natural to expect that its corresponding surface integral would also be identically zero. As in the case of Green's Theorem, one can use Stokes' Theorem to replace a line integral

by a surface integral to remove conservative vector fields that may complicate the computation without affecting the value of the integral.

Although we will see that surface integrals usually depend on the surface, the surface integrals in Stokes' Theorem give the same value for integrals over any surfaces having the same boundary curve.

Exercises

#3. $\mathbf{F} = \left\langle x^2 e^{yz}, y^2 e^x z, z^2 e^{xy} \right\rangle$, $\mathcal{C}$ is boundary of part of $x^2 + y^2 + z^2 = 4$ where $z \geq 0$.

#7. $\mathbf{F} = \left\langle x + y^2, y + z^2, z + x^2 \right\rangle$ $\mathcal{C}$ is triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$.

#9. $\mathbf{F} = \langle 2z, 4x, 5y \rangle$, $\mathcal{C}$ is intersection of $z = x + 4$ with $x^2 + y^2 = 4$.