The Divergence Theorem. The proof of the divergence is a three dimensional version of the proof of Green's theorem. It is proved first for regions for which all sections parallel to one of the coordinate axes are intervals (or empty). Evaluating a triple integral of an expression f over a solid body $\mathcal{B}$ as an iterated integral begins by using the fundamental theorem of one-variable calculus to find an expression R such that $\partial R/\partial z = f$. This is evaluated as integral over the projection in the xy plane of the difference of R for z at the top and bottom of $\mathcal{B}$. However, the operation of substituting the value of z on a surface and integrating R with respect to x and y is exactly the same as the surface integral of $R\mathbf{k}$ over the surface. Any part of the surface perpendicular to the xy plane gives a zero contribution to such a surface integral. The outward normal to the boundary $\mathcal{S}$ of $\mathcal{B}$, normalized to give an integral with respect to area in the xy plane has third component $+1$ on the top and third component -1 on the bottom. Since we have chosen a vector field whose first two components are zero, it is only the third component of the normal to the surface that contributes to the integral.

In the same way, on can integrate first with respect to x or y. This gives the three parts of the divergence theorem:

$$\iint_{\mathcal{S}} \langle P, 0, 0\rangle \cdot \mathbf{n}\, dS = \iiint_{\mathcal{B}} \frac{\partial P}{\partial x}\, dV$$
$$\iint_{\mathcal{S}} \langle 0, Q, 0\rangle \cdot \mathbf{n}\, dS = \iiint_{\mathcal{B}} \frac{\partial Q}{\partial y}\, dV$$
$$\iint_{\mathcal{S}} \langle 0, 0, R\rangle \cdot \mathbf{n}\, dS = \iiint_{\mathcal{B}} \frac{\partial R}{\partial z}\, dV$$

Adding these shows that the the integral of a vector field over $\mathcal{S}$ is equal to the integral of its divergence over $\mathcal{B}$.

The divergence theorem may be extended to more general regions that can by cut into pieces having the form used in the proof. Since the surface integral is oriented, its value on a cut will be counted with opposite signs when considered as part of the boundary of regions on the two sides of the cut. When one adds over the dissection of the region, the contribution of each cut simplifies to zero.

How is the divergence theorem used? Most exercises in the use of the divergence theorem, as is also true of the other theorems of vector calculus, use the triple integral to evaluate the surface integral. Since this is opposite to the direction of the proof, some explanation should be given. There are several reasons for this.

(1) It is certainly true, as in the proof of the divergence theorem, that the first step in the evaluation of a triple iterated integral over $\mathcal{B}$ can be interpreted as a surface integral over the boundary of $\mathcal{B}$. However, what the calculation actually gives is a double integral over the projection of $\mathcal{B}$ into one of the coordinate planes, which includes the use of a parameterization to begin the computation of the surface integral.

(2) The different components of the surface integral appear in the proof of the divergence theorem as the results of using different approaches to setting up an integral over $\mathcal{B}$ as in iterated integral. Normally, one only writes the integral in the one form that will be easiest to evaluate. When different terms of the triple integral are evaluated using different orders of inte-

gration, you are essentially choosing to assign those terms to different parts of the divergence.

(3) When the integrand in the surface integral is expressed in terms of polynomials, it becomes simpler when differentiated. Thus the integrand of the triple integral is simpler than that of the surface integral.

(4) Changes to cylindrical or spherical coordinates could be explained by using those coordinate systems to parameterize surfaces in surfaces integrals, but a direct description of triple integrals in these coordinate systems is often easier in those case where such coordinate systems are relevant.

(5) Certain triple integrals are known because they represent volumes or moments that are remembered from previous computations. It is not necessary to repeat these computations in order to get the answer, and the coordinate system of the current problem may make those computations tedious. Since the goal of these calculations is to get correct answers, some effort should be made to formulate problems in a way that simplifies computation.

Exercises 16.9 We will ignore specific instructions in the textbook and consider both sides of the divergence theorem identity for all examples. Typically, the vector field $\mathbf{F}$ to be integrated on the surface will be given, but the surface will be described only as the outwardly oriented boundary of a solid region.

#3. $\mathbf{F} = \langle 3x, xy, 2xz \rangle$. Region is cube in which each coordinate is between 0 and 1.

#5. $\mathbf{F} = \langle xy, yz, zx \rangle$. Region is solid cylinder defined by $x^2 + y^2 \leq 1$ and $0 \leq z \leq 1$.

#7. $\mathbf{F} = \left\langle 3x^2y^3, 9x^2yz^2, -4xy^2 \right\rangle$. Region is cube with vertices $(\pm 1, \pm 1, \pm 1)$.

#11. $\mathbf{F} = \left\langle 3xy^2, xe^z, z^3 \right\rangle$. Region is the cylinder bounded by $y^2 + z^2 = 1$, $x = -1$, $x = 2$.