Why "divergence"? The divergence theorem gives a physical interpretation of the divergence of a vector field. The surface integral of the vector field over any closed surface is equal to the integral of the divergence of the vector field over the body bounded by that surface. If the body is a small spherical ball, then the latter integral is equal to the volume of the ball times the average value of the divergence in the ball.

In the fluid flow model, the surface integral represents the amount of fluid passing through the boundary of the sphere. For this to be positive, it must be created in some way inside the body. If there are no sources, then the fluid must be expanding. In this case, that expansion is measured by the divergence.

Change of variables. If the x and y coordinates of our favorite plane are expressed in terms of another pair of variables u and v, we would like to know how to use these new variables to compute integrals. We have seen the example of polar coordinates. By repeating the construction of the integral from Riemann sums, using *polar rectangles* instead of ordinary rectangles,

we found that

$$dA = dx\,dy = r\,dr\,d\theta.$$

Lurking behind this argument was a belief that area had a number of nice properties that we were prepared to take as axioms. Some of these have been proved in the course of doing some exercises in calculus. In particular, we really do know that polar rectangles have the expected area when computed using Riemann sums based on ordinary rectangles, and *vice versa*.

To do this more generally, u and v should be taken as rectangular coordinates in a plane of their own. In nice cases, the equations defining x and y can be solved for u and v giving a well-defined region in this uv plane corresponding to any given region in the xy plane. In the most familiar case of polar coordinates, it is necessary to require $r > 0$ and $0 \le \theta < 2\pi$ to get a unique choice of polar coordinates corresponding to given rectangular coordinates other than $(0, 0)$. The origin plays a special role, as a single point in the xy plane represented by polar coordinates with $r = 0$ and

any value of θ. Since we are concerned with formulas for area, this causes no difficulty since lines have area zero.

Computation with coordinates u and v can be described in terms of integrals in the uv plane. To interpret results in the xy plane, we construct the sets that are the images of the rectangles used for Riemann sums in the uv plane. This was what was done when we spoke of polar rectangles — they really were rectangles in an $r\theta$ plane that we never described. Since the xy plane is supposed to describe reality, every area in the uv plane must be multiplied by a factor giving the ratio of the area of its image in the xy plane to the area seen in the uv plane before being used in a Riemann sum. Since the Riemann sums are only a technical device which leads to the integral by finding a limit, it is the limit of the ratio of areas that is needed. Some technical analysis is needed to make sure that all integrals exist and the ratios behave in the way that we expect, but the picture used in this outline illustrates the change-of-variables formula in the context of the definition of the integral.

In the limit, the image of a rectangle with sides of length du and dv based at a point (u, v) in the uv plane will look like a parallelogram based at (x, y) whose sides are the vectors

$$\langle x_u, y_u\rangle du \text{ and } \langle x_v, y_v\rangle dv.$$

The quantity $J = x_u y_v - x_v y_u$ measures the ratio of areas (with the sign giving the relation between the orientation of the two figures). For polar coordinates,

$$\begin{pmatrix} x_r & x_\theta \\ y_r & y_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix},$$

so

$$J = r\cos^2\theta + r\sin^2\theta = r.$$

The quantity J is called the **Jacobian** of the function expressing x and y in terms of u and v.

The full rule for changing coordinates in a double integral calls for using the given expressions for x and y to replace all appearances of these variables in the integrand with expressions in u and v, and replacing the expression $dx\,dy$ by $J\,du\,dv$. In general, J will be an expression in u and v.

Relation to Green's theorem. Another proof of this formula can be given by finding the effect of changes of coordinates on line integrals. The proof of Green's theorem shows that every double integral in the xy can be evaluated as a suitable line integral around its boundary. This is converted to a line integral in the uv plane. This line integral is converted by Green's theorem into a double integral in the uv plane. In this proof, we can start with a line integral

$$I = \oint P\,dx + Q\,dy.$$

Here, P and Q will be functions of x and y. We then express x and y in terms of u and v, which will be expressed in terms of u and v by composition with the functions giving x and y in terms of u and v. This substitution will express I as a line integral in the uv-plane. In addition to obtaining P and Q in terms of u

and v, we need to write

$$dx = x_u\,du + x_v\,dv,$$
$$dy = y_u\,du + y_v\,dv.$$

Thus,

$$I = \oint (Px_u + Qy_u)\,du + (Px_v + Qy_v)\,dv.$$

Applying Green's theorem in the uv-plane gives an integrand of the double integral that is

$$\begin{aligned}(Px_v + Qy_v)_u - (Px_u + Qy_u)_v = \\ Px_{vu} + Qy_{vu} + (P_x x_u + P_y y_u)x_v \\ +(Q_x x_u + Q_y y_u)y_v \\ -Px_{uv} - Qy_{uv} - (P_x x_v + P_y y_v)x_u \\ -(Q_x x_v + Q_y y_v)y_u \\ = (Q_x - P_y)(x_u y_v - x_v y_u)\end{aligned}$$

Here the first part is the integrand of the original double integral in the xy plane composed with the definition of x and y in terms of u and v, and the second

factor is the expression that our previous analysis has shown us to expect.

Surface integrals. To evaluate a surface integral on the graph of a function $z = f(x, y)$, one writes

$$d\mathbf{S} = \langle -f_x, -f_y, 1\rangle\,dx\,dy$$

to express the integral in the upward orientation. If the surface has a parametric description, then x, y and z **on the surface** are functions of u and v. Making the assumption that the surface can be given as the graph of a function allows the integral to be written as an integral over the projection into the xy-plane. Since this is the projection of a region on the surface, x and y have exactly the same meaning as they have on the surface. In the resulting integral, the expression of x and y in terms of u and v may be made. Although the function giving z in terms of x and y may not be known explicitly, when we compose the expression for z in terms of x and y with the definitions of these variables in terms of u and v, we get the **given** expression for z in terms of u and v. When a normal to the surface is scaled so that the product with $du\,dy$ is $d\mathbf{S}$, the third

component must be the Jacobian of the expression of x and y in terms of u and v. This scaling is exactly the one given by the cross product of tangents to the grid curves.

Typically, only a small part of the surface can be considered to be the graph of a function, but there is nothing special about z to make it the dependent variable. The same approach applies whenever one of the variables is a function of the other two, and any general formula proved under these assumptions will be valid whenever the surface can be written as a union of pieces such that each piece has one of the variables as a function of the other two. This covers essentially all surfaces. The divergence theorem then allows what we know about changes of variable in double integrals to be applied to triple integrals. The result will express the Jacobian as the determinant of the 3 by 3 matrix of partial derivatives. Since this determinant gives the ration of volumes of small parallelepipeds in the two coordinate systems, the result is exactly what is expected. This can be used to develop a theory of integration in Euclidean spaces of any number of dimensions, where an n dimensional space is described by *hypothesizing* the existence of n mutually perpendicular vectors.

Exercises. Give the Jacobians for each of the following changes of variable.

#1. $x = u + 4v$, $y = 3u - 2v$.

#3. $x = \dfrac{u}{u+v}$, $y = \dfrac{v}{u-v}$.

#5. $x = uv$, $y = vw$, $z = uw$.

#9. Find the image of the triangle with vertices $(0, 0)$, $(1, 1)$, $(0, 1)$ in the uv-plane under $x = u^2$, $y = v$.

Coordinate systems in Maple. Maple has the ability to work with curves described in many different coordinate systems. This includes polar coordinates and its three dimensional relatives, but also some that never make it into basic textbooks. Since these coordinate systems are actually used somewhere, they make more interesting examples for computing Jacobians, as well as for studying the nature of the grid curves in the space with rectangular coordinates x

and y (and z for three dimensional examples). Although the general methods can compute with any $\langle x(u, v), y(u, v)\rangle$, the useful examples often have grid curves that are mutually perpendicular (as in polar coordinates). In this case, the Jacobian is simple the product of the lengths of the tangent vectors to the grid curves. Sometimes there are constant factors in the formulas that seem arbitrary, but have their roots in some general methods for producing such functions and lead to more pleasing geometric properties. Although we are concerned here only with the Jacobian determinant, Linear Algebra allows one to interpret the whole matrix of partial derivatives. Indeed, is is that algebra of that matrix that expresses the **chain rule** for the change of coordinates. For the example, we write the matrix, although you may choose to see only its determinant.

We give some two dimensional examples. There are many more, as well as a wealth of three dimensional examples.

Parabolic coordinates.

$$x = \tfrac{1}{2}(u^2 - v^2)$$
$$y = uv$$

The matrix of partial derivatives is

$$\begin{pmatrix} u & -v \\ v & u \end{pmatrix}$$

Hyperbolic coordinates.

$$x = \sqrt{\sqrt{u^2 + v^2} + u}$$
$$y = \sqrt{\sqrt{u^2 + v^2} - u}$$

The matrix of partial derivatives is (using x and y as abbreviations for their expressions in terms of u and v)

$$\begin{pmatrix} \frac{x}{2\sqrt{u^2+v^2}} & \frac{v}{2x\sqrt{u^2+v^2}} \\ \frac{-y}{2\sqrt{u^2+v^2}} & \frac{v}{2y\sqrt{u^2+v^2}} \end{pmatrix}$$

Notice that $x^2y^2 = v^2$, which can be helpful in recognizing the grid curves and inverting this change of variables.

Elliptic coordinates.

$$x = \cosh u \cos v$$
$$y = \sinh u \sin v$$

The matrix of partial derivatives is

$$\begin{pmatrix} \sinh u \cos v & -\cosh u \sin v \\ \cosh u \sin v & \sinh u \cos v \end{pmatrix}$$