October 23, 2001

Find and classify all the critical points of the function

$$f(x,y) = x^2y^3 + 2xy - y.$$

**Solution:** The critical points are obtained as the simultaneous solutions of

$$\begin{cases} f_x(x,y) = 2xy^3 + 2y = 0 \\ f_y(x,y) = 3x^2y^2 + 2x - 1 = 0 \end{cases}$$

The first equation is $2y(xy^2+1) = 0$, so either $y = 0$ or $xy^2 + 1 = 0$.

Case 1: $y = 0$. Then $x = 1/2$ from the second equation, so $(1/2, 0)$ is a critical point.

Case 2: $xy^2 + 1 = 0$, so $x = -y^{-2}$. Substituting in the second equation gives $3y^{-4}y^2 - 2y^{-2} = 1$, so $y^{-2} = 1$, $y^2 = 1$, $y = \pm 1$, and $x = -y^{-2} = -1$. So there are two more critical points $(-1, \pm 1)$. (Even easier is to use $xy^2 + 1 = 0$ in the form $y^2 = -x^{-1}$, substituting in the second equation giving $-3x + 2x - 1 = 0$, $x = -1$, $y^2 = -x^{-1} = 1$, $y = \pm 1$.)

We found 3 critical points: $(1/2, 0)$, $(-1, 1)$ and $(-1, -1)$.

Now apply the second derivative test.

$A = f_{xx}(x,y) = 2y^3$, $B = f_{xy}(x,y) = 6xy^2 + 2$ and $C = f_{yy}(x,y) = 6x^2y$.

At $(1/2, 0)$, $A = 0$, $B = 2$ and $C = 0$ so $AC - B^2 = -4 < 0$.
Conclusion: $f$ has a saddle point at $(1/2, 0)$.

At $(-1, 1)$, $A = 2$, $B = -4$ and $C = 6$. $AC - B^2 = -4 < 0$.
Conclusion: $f$ has a saddle point at $(-1, 1)$.

At $(-1, -1)$, $A = -2$, $B = -4$ and $C = -6$. $AC - B^2 < 0$ again.
Conclusion: $f$ has a saddle point at $(-1, -1)$.

October 23, 2001

Find and classify all the critical points of the function

$$f(x, y) = x^3y^2 - 2xy - x.$$

**Solution:** The critical points are obtained as the simultaneous solutions of

$$\begin{cases} f_x(x, y) = 3x^2y^2 - 2y - 1 = 0 \\ f_y(x, y) = 2x^3y - 2x = 0 \end{cases}$$

The second equation is $2x(x^2y - 1) = 0$, so either $x = 0$ or $x^2y - 1 = 0$.

Case 1: $x = 0$. Then $y = -1/2$ from the first equation, so $(0, -1/2)$ is a critical point.

Case 2: $x^2y - 1 = 0$, so $y = x^{-2}$. Substituting in the first equation gives $3x^2x^{-4} - 2x^{-2} = 1$, so $x^{-2} = 1$, $x^2 = 1$, $x = \pm 1$, and $y = x^{-2} = 1$. So there are two more critical points $(\pm 1, 1)$. (Even easier is to use $x^2y - 1 = 0$ in the form $x^2 = y^{-1}$, substituting in the first equation giving $3y - 2y - 1 = 0$, $y = 1$, and then $x^2 = y^{-1} = 1$, $x = \pm 1$.)

We found 3 critical points: $(0, -1/2)$, $(1, 1)$ and $(-1, 1)$.

Now apply the second derivative test.

$A = f_{xx}(x, y) = 6xy^2$, $B = f_{xy}(x, y) = 6x^2y - 2$ and $C = f_{yy}(x, y) = 2x^3$.
At $(0, -1/2)$, $A = 0$, $B = -2$ and $C = 0$ so $AC - B^2 = -4 < 0$.
Conclusion: $f$ has a saddle point at $(0, -1/2)$.

At $(1, 1)$, $A = 6$, $B = 4$ and $C = 2$. $AC - B^2 = -4 < 0$.
Conclusion: $f$ has a saddle point at $(1, 1)$.

At $(-1, 1)$, $A = -6$, $B = 4$ and $C = -2$. $AC - B^2 < 0$ again.
Conclusion: $f$ has a saddle point at $(-1, 1)$.