Math 251–13–14–15 (`CIRCLE YOUR SECTION`) SOLUTIONS, version 1

November 29, 2001

1. Compute $\int_C (5y+e^x)\,dx+(2x+e^y)\,dy$ where C is the circle of radius 5 centered at $(13,57)$, traversed counter-clockwise. (Hint: Green's Theorem)

Solution: Let R be the disk whose circumference is C. By Green's Theorem

$$\begin{aligned}\int_C (5y+e^x)\,dx+(2x+e^y)\,dy &= \iint\limits_R \left(\frac{\partial(2x+e^y)}{\partial x}-\frac{\partial(5y+e^x)}{\partial y}\right)\,dA\\ &= \iint\limits_R (2-5)\,dA = -3\iint\limits_R\,dA\\ &= -3\cdot(\text{area of R}) = -3\cdot 25\pi = -75\pi.\end{aligned}$$

2. Let $\mathbf{F}(x,y)=(x^3+6x^2y^2)\mathbf{i}+(2y+4x^3y)\mathbf{j}$. Either find a function $f(x,y)$ such that $\mathbf{F}=\nabla f$, or explain why such a function f does not exist.

Solution: First,

$$\frac{\partial(x^3+6x^2y^2)}{\partial y}=12x^2y=\frac{\partial(2y+4x^3y)}{\partial x}$$

so f might exist, and indeed does exist since $\mathbf{F}$ has no singularities, i.e. is defined and has derivatives of all orders at each point of the xy-plane. (This step is not necessary in this example, strictly speaking, but is highly advisable. If it had turned out that $P_y\neq Q_x$, then it would be futile to try to find f.)
Next,

$$f(x,y)=\int (x^3+6x^2y^2)\,dx=\frac{x^4}{4}+2x^3y^2+g(y)$$

where

$$2y+4x^3y=\frac{\partial f}{\partial y}=0+4x^3y+g'(y)$$

so $g'(y)=2y,\ g(y)=\int 2y\,dy=y^2+C$ works.
Answer: $f(x,y)=\dfrac{x^4}{4}+2x^3y^2+y^2.$