1. Find the points on the plane $z = x + y + 1$ that are closest to the origin, without using Lagrange multipliers.

Solution: The function to be minimized is the distance $d = \sqrt{x^2 + y^2 + z^2}$ from a point (x, y, z) on the plane to the origin. Since the point is on the plane, $z = x+y+1$ so we are trying to minimize

$$f(x, y) = x^2 + y^2 + (x + y + 1)^2.$$

To find the minimum we seek the critical points:

$$f_x = 2x + 2(x + y + 1) = 0 \text{ and } f_y = 2y + 2(x + y + 1) = 0.$$

The first equation gives $y = -2x - 1$, and substituting this in the second equation gives $-2x - 1 + x - 2x - 1 + 1 = 0$, so $x = -1/3$, $y = -2x - 1 = -1/3$. The only critical point is $(-1/3, -1/3)$, so it must give the closest point to the origin. That closest point satisfies $z = x + y + 1 = 1/3$ so is the point $(-1/3, -1/3, 1/3)$.

2. Use Lagrange multipliers to do the same problem.

Solution: We are to minimize the objective function d^2, that is,

$$f(x, y, z) = x^2 + y^2 + z^2,$$

subject to the constraint

$$g(x, y, z) = x + y - z + 1 = 0.$$

The Lagrange multiplier equations are $\nabla f = \lambda \nabla g$ and the constraint equation, namely

$$\begin{aligned} 2x &= \lambda \cdot 1 \\ 2y &= \lambda \cdot 1 \\ 2z &= \lambda \cdot (-1) \\ x + y - z + 1 &= 0 \end{aligned}$$

The first three equations give $x = y = -z$, and then the third equation gives $3x = 3y = -3z = -1$, so $(x, y, z) = (-1/3, -1/3, 1/3)$.