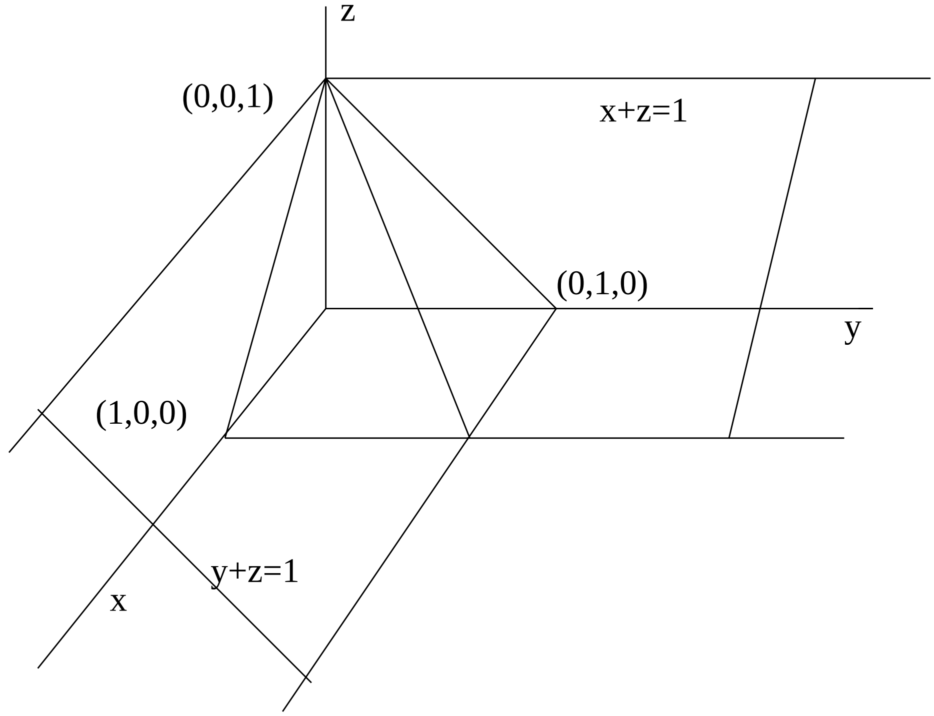
Math 251–13–14–15 November 7, 2001 SOLUTIONS

1. Evaluate the triple integral $\iiint\limits_E xy\,dV$ where E is bounded by the planes $x=0$, $y=0$, $z=0$, $y+z=1$ and $x+z=1$.



Solution:
Lines parallel to the z-axis exit E on one of the top surfaces $y+z=1$ or $x+z=1$. Since there are two "pieces" to the top surface it is awkward to evaluate the integral as type I. Use type II instead: lines parallel to the y-axis enter E through the xz-plane ($y=0$) and exit through the plane $y+z=1$ ($y=1-z$). These give the innermost limits of integration. The shadow region in the xz plane is the left triangle with vertices $(0,0,0)$, $(0,0,1)$ and $(1,0,0)$, bounded by the x and z axes and the line $x+z=1$ ($z=1-x$). So

$$\iiint\limits_E xy\,dV = \int_{x=0}^{1}\int_{z=0}^{1-x}\int_{y=0}^{1-z} xy\,dy\,dz\,dx = \int_{x=0}^{1}\int_{z=0}^{1-x} \frac{xy^2}{2}\Big|_{y=0}^{y=1-z}\,dz\,dx$$

$$= \int_{x=0}^{1}\int_{z=0}^{1-x} \frac{x(1-z)^2}{2}\,dz\,dx$$

$$= \int_{x=0}^{1} \frac{-x(1-z)^3}{6}\Big|_{z=0}^{z=1-x}\,dx = \int_{x=0}^{1} \frac{-x^4}{6}+\frac{x}{6}\,dx$$

$$= \frac{-x^5}{30}+\frac{x^2}{12}\Big|_0^1 = \frac{1}{12}-\frac{1}{30}=\frac{1}{20}.$$