REVIEW PROBLEMS FOR FINAL EXAM 251-13-14-15, FALL 2001

The exam will cover the entire syllabus. You will be allowed to use a calculator without symbolic capabilities, such as TI-82, 83, 85, 86. You may not bring a crib sheet. You will be given a short list of formulas with the exam (see next page).
The following problems are mostly taken or adapted from recent 251 exams. (Answers are not guaranteed.)

1. Let C be the triangle in the plane with vertices $(0,1)$, $(1,0)$ and $(1,1)$, oriented counter-clockwise.

(a) Evaluate $\oint_C 2y^2\,dx + 2x\,dy$ directly as a line integral.

(b) Evaluate the same line integral by using Green's Theorem. Ans. $-1/3$

2. Use the curl operator to show that the field

$$\boldsymbol{F}(x,y,z) = (3x^2y^2z+1)\,\boldsymbol{i} + (2x^3yz+2)\,\boldsymbol{j} + (x^3y^2+3)\,\boldsymbol{k}$$

is conservative. Then find a function f such that $\nabla f = \boldsymbol{F}$. Then quickly calculate $\int_C \boldsymbol{F} \bullet d\boldsymbol{r}$ where C is the path $\boldsymbol{r}(t) = t^{10}\,\boldsymbol{i} + \arctan t\,\boldsymbol{j} + \ln(t+1)\,\boldsymbol{k}$, $0 \le t \le 1$. Ans. $\frac{\pi^2}{16}\ln 2 + 1 + \frac{\pi}{2} + 3\ln 2$

3. Let E be the solid region above the plane $z = 3$ and below the paraboloid $z = 4 - x^2 - y^2$. Let Σ be the boundary surface of E. Let $\boldsymbol{F}(x,y,z) = (y+x)\,\boldsymbol{i} + (y-x)\,\boldsymbol{j}$. Verify the Divergence Theorem

$$\iiint_E \operatorname{div}\boldsymbol{F}\,dV = \iint_\Sigma \boldsymbol{F} \bullet \boldsymbol{n}\,dS$$

by computing both sides separately and showing that they are equal. Ans. $\pi = \pi$

4. Use the Divergence Theorem to calculate $\iint_\Sigma \boldsymbol{F} \bullet \boldsymbol{n}\,dS$ where Σ is the (entire) surface of the cylinder enclosed by the surfaces $x^2 + y^2 = 1$, $z = 1$ and $z = 3$, and $\boldsymbol{F} = (x^3 + 2y)\,\boldsymbol{i} + (2x + y^3)\,\boldsymbol{j} + 3z\,\boldsymbol{k}$.
Ans. (Corrected) 9π

5. Find the surface area of that portion of the surface $z = x^2 + y^2 + 9$ lying between the planes $z = 10$ and $z = 13$.
Ans. $\pi(17^{3/2} - 5^{3/2})/6$

6. Use Green's Theorem to calculate the work done by the force field $\boldsymbol{F} = (x - y^2)\,\boldsymbol{i} + (x^2 - y)\,\boldsymbol{j}$ on a particle moving in straight line segments from $(2,1)$ to $(2,4)$ to $(3,4)$ to $(3,1)$ and back to $(2,1)$.
Ans. (Corrected) -30

7. Let $\boldsymbol{F}(x,y,z) = (x+y)\,\boldsymbol{i} + (y-x)\,\boldsymbol{j} + z\,\boldsymbol{k}$.

(a) Compute $\oint_C \boldsymbol{F} \bullet d\boldsymbol{r}$ where C is the circle $x^2 + y^2 = 4$ in the xy-plane, oriented clockwise.

(b) Compute $\iint_\Sigma \operatorname{curl}\boldsymbol{F} \bullet \boldsymbol{n}\,dS$ where Σ is the upper half of the sphere $x^2 + y^2 + z^2 = 4$, and $\boldsymbol{n}$ points up.

(c) Reconcile your answers to (a) and (b) with Stokes' Theorem. Ans. (a) 8π (b) -8π.

8. Let Σ_1 be the disk $x^2 + y^2 \le 1$ in the xy-plane, oriented with $\boldsymbol{n}$ pointing up. Let Σ_2 be the "northern hemisphere" $x^2 + y^2 + z^2 = 1$, $z \ge 0$, also oriented with $\boldsymbol{n}$ pointing up. Let $\boldsymbol{F}$ be any twice differentiable vector field. Give two explanations why

$$\iint_{\Sigma_1} \operatorname{curl}\boldsymbol{F} \bullet \boldsymbol{n}\,dS = \iint_{\Sigma_2} \operatorname{curl}\boldsymbol{F} \bullet \boldsymbol{n}\,dS$$

(a) one using Stokes' Theorem

(b) the other using the Divergence Theorem (first show that $\operatorname{div}\operatorname{curl}\boldsymbol{F} = 0$).