1. If $f(x,y) = xy^2 + 4x^3y^5$, find f_x, f_{xy} and f_{xx}.

Solution: $f_x(x,y) = y^2 + 12x^2y^5$, $f_{xy}(x,y) = 2y + 60x^2y^4$ *and* $f_{xx}(x,y) = 24xy^5$.

2. Consider the triangle with vertices $P(1,2,3)$, $Q(2,-2,3)$ and $R(0,3,1)$.

(a) Find the angle at vertex Q.

Solution: $\overrightarrow{QP} = \langle -1,4,0\rangle$ *and* $\overrightarrow{QR} = \langle -2,5,-2\rangle$. *The angle is* $\arccos \frac{\overrightarrow{QP}\bullet\overrightarrow{QR}}{|\overrightarrow{QP}||\overrightarrow{QR}|} = \arccos 22/\sqrt{17}\sqrt{33}$.

(b) Find an equation of the plane in which this triangle lies.

Solution: *Using the normal vector* $\overrightarrow{N} = \overrightarrow{QP} \times \overrightarrow{QR} = -8\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ *and the point* P *we get the equation* $-8(x-1) - 2(y-2) + 3(z-3) = 0$. *There are many correct answers.*

(c) Find the area of the triangle.

Solution: *The area is* $(1/2)|\overrightarrow{QP} \times \overrightarrow{QR}| = \sqrt{77}/2$.

3. A moving particle has position function $\mathbf{r}(t) = (t^3/3)\mathbf{i} + t^2\mathbf{j} + 2t\mathbf{k}$. Find

(a) the distance traveled between times $t = 1$ and $t = 3$

Solution: $\mathbf{r}'(t) = t^2\mathbf{i} + 2t\mathbf{j} + 2\mathbf{k}$ *and* $|\mathbf{r}'(t)| = \sqrt{t^4 + 4t^2 + 4} = \sqrt{(t^2+2)^2} = t^2 + 2$ *so the distance is* $\int_1^3 |\mathbf{r}'(t)|\,dt = \int_1^3 (t^2+2)\,dt = 38/3$.

(b) The normal component of acceleration at time $t = 1$

Solution: *From (a),* $v(t) = t^2 + 2$ *so* $v(1) = 3$. *Then* $a_{nor}(1) = \kappa(1)v(1)^2 = 9\kappa(1)$. *Using (d) below,* $a_{nor}(1) = 2$.

(c) the unit tangent vector at time $t = 1$

Solution: $\mathbf{T}(1) = \frac{1}{|\mathbf{r}'(1)|}\mathbf{r}'(1) = \frac{1}{3}\langle 1,2,2\rangle$.

(d) the curvature of the path at time $t = 1$

Solution: $\mathbf{r}''(t) = \langle 2t,2,0\rangle$, *so* $\mathbf{r}'(1) \times \mathbf{r}''(1) = \langle 1,2,2\rangle \times \langle 2,2,0\rangle = -4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ *and* $\kappa(1) = \frac{|\langle 1,2,2\rangle \times \langle 2,2,0\rangle|}{3^3} = \frac{6}{27} = \frac{2}{9}$.

4. Find an equation of the tangent plane to the surface $z = \frac{12x}{x^2 - y^2}$ at the point $(2,1,8)$.

Solution: $z_x = \frac{12(x^2-y^2) - 2x(12x)}{(x^2-y^2)^2} = \frac{-12(x^2+y^2)}{(x^2-y^2)^2}$ *and* $z_y = \frac{-(-2y)(12x)}{(x^2-y^2)^2} = \frac{24xy}{(x^2-y^2)^2}$. *Then* $z_x\Big|_{(2,1,8)} = (-60/9)$ *and* $z_y\Big|_{(2,1,8)} = (48/9)$. *The tangent plane is*

$$z - 8 = (-60/9)(x-2) + (48/9)(y-1).$$

5. If $z = f(x, y)$, $x = r^3 s$ and $y = rs^3$, express $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial z}{\partial s}$ in terms of $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$, r and s.

Solution: $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial r} = \dfrac{\partial z}{\partial x}3r^2 s + \dfrac{\partial z}{\partial y}s^3$ *and*
$\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s} = \dfrac{\partial z}{\partial x}r^3 + \dfrac{\partial z}{\partial y}3rs^2.$

6. If $x^2 y^3 z = xy^2 z^3 + 6$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$. **Solution:** *By implicit differentiation (with respect to x, holding y fixed):* $2xy^3 z + x^2 y^3 \dfrac{\partial z}{\partial x} = y^2 z^3 + xy^2 \cdot 3z^2 \dfrac{\partial z}{\partial x}$ *and so*
$\dfrac{\partial z}{\partial x} = \dfrac{-2xy^3 z + y^2 z^3}{x^2 y^3 - 3xy^2 z^2}$. *Alternatively, rewrite the equation as $F(x, y, z) = 0$ where $F(x, y, z) = x^2 y^3 z - xy^2 z^3 - 6$, and use the formula*

$$\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z} = \frac{-2xy^3 z + y^2 z^3}{x^2 y^3 - 3xy^2 z^2}.$$

Similarly (by either method) $\dfrac{\partial z}{\partial y} = \dfrac{-3x^2 y^2 z + 2xyz^3}{x^2 y^3 - 3xy^2 z^2}$. *The answers can be simplified.*

7. (a) Find an equation of a plane which is parallel to both of the lines

$$\begin{cases} x(t) = 2 - t \\ y(t) = 3 + 2t \\ z(t) = 4 + t \end{cases} \quad \text{and} \quad \begin{cases} x(t) = 1 + t \\ y(t) = 2 - 5t \\ z(t) = 4 - 2t \end{cases}$$

Solution: *For any such plane we can use the normal vector $\langle -1, 2, 1\rangle \times \langle 1, -5, -2\rangle = \langle 1, -1, 3\rangle$. Using the point $(2, 3, 4)$ (from the first line) gives the plane $(x-2) - (y-3) + 3(z-4) = 0$. (Any point could be used but this helps to answer (b).)*

(b) Do these two lines intersect or not? Justify your answer.

Solution: *Yes, they intersect. Substituting the $x(t)$, $y(t)$ and $z(t)$ for either line into the equation of the plane in (a) yields $0 = 0$, so both lines lie in this plane. As the two lines lie in a plane and are not parallel, they do intersect. (It is not enough just to say that they are not parallel; in three dimensions, two lines can be skew.) Alternate solution: the lines intersect if and only if the three equations $2 - t = 1 + t'$, $3 + 2t = 2 - 5t'$ and $4 + t = 4 - 2t'$ have a common solution t, t'. Solving this system of three equations yields the solution $t = 2$, $t' = -1$ (check it!) so the lines do intersect.*

8. Sketch several (at least 3) level curves in the x, y-plane of the function $f(x, y) = 2x^2 + y$, and find an equation of the level curve passing through the point $(3, 2)$.

Solution:
The level curves are the parabolas $2x^2 + y = k$ for various constants k, such as $2x^2 + y = 0$, $2x^2 + y = 1$, $2x^2 + y = 2$, etc. The level curve passing through the point $(3, 2)$ has equation $2x^2 + y = 2 \cdot 3^2 + 2$, that is, $2x^2 + y = 20$.