(15) 1. Consider the directional derivatives at $(2,1)$ of $f(x,y) = xy^{-1} + x^2y$.

(a) Find the directional derivative in the direction from $(2,1)$ toward the point $(-1,4)$.

Solution: $\nabla f = (y^{-1} + 2xy)\mathbf{i} + (-xy^{-2} + x^2)\mathbf{j}$ *so* $\nabla f(2,1) = 5\mathbf{i} + 2\mathbf{j}$. *Also a unit vector in the direction from* $(2,1)$ *toward* $(-1,4)$ *is* $\mathbf{u} = \dfrac{1}{\sqrt{18}}(-3\mathbf{i} + 3\mathbf{j})$. *The required directional derivative is* $D_{\mathbf{u}}f(2,1) = \nabla f(2,1) \bullet \mathbf{u} =$
$(5\mathbf{i} + 2\mathbf{j}) \bullet \dfrac{1}{\sqrt{18}}(-3\mathbf{i} + 3\mathbf{j}) = -\dfrac{3}{\sqrt{2}}$.

(b) Of all the directional derivatives of f at $(2,1)$, what is the value of the largest one, and what direction gives rise to it?

Solution: *The largest value is* $\left|\nabla f(2,1)\right| = |5\mathbf{i} + 2\mathbf{j}| = \sqrt{29}$. *This occurs in the direction of* $\nabla f(2,1) = 5\mathbf{i} + 2\mathbf{j}$.

(c) What is the slope (at $(2,1)$) of the level curve of f passing through $(2,1)$?

Solution: *The level curve is perpendicular to* $\nabla f(2,1) = 5\mathbf{i} + 2\mathbf{j}$. *Hence its slope is* $-5/2$.

(10) 2. Let C be the straight line segment from $(0,1)$ to $(4,3)$. Find $\displaystyle\int_C x^2y\,ds$.

Solution: *One parametrization for* C *uses the starting point* $(0,1)$ *and the direction vector* $\mathbf{D} = 4\mathbf{i} + 2\mathbf{j}$*: thus a parametrization is*

$$x = 4t,\ y = 1{+}2t,\ t: 0 \to 1,\ \textit{whence}\ \frac{dx}{dt} = 4,\ \frac{dy}{dt} = 2,\ ds = \left(4^2 + 2^2\right)^{1/2} = 2\sqrt{5}$$

Thus

$$\int_C x^2y\,ds = \int_0^1 (4t)^2(1{+}2t)\cdot 2\sqrt{5}\,dt = 32\sqrt{5}\int_0^1 t^2(1{+}2t)\,dt = 32\sqrt{5}\cdot\frac{5}{6} = \frac{80\sqrt{5}}{3}.$$

(10) 3. Let $f(x,y) = x^3 - 6xy + y^2$. Find all the critical points of f, and classify them as local maxima, local minima, or saddle points.

Solution: *Critical points are the solutions of* $f_x = f_y = 0$, *that is,*
$3x^2 - 6y = 2y - 6x = 0,\ x^2 = 2y = 6x,\ x = 0$ *or* $6,\ y = 3x = 0$ *or* 18. *So there are two critical points:* $(x,y) = (0,0)$ *or* $(6,18)$.
Let $A = f_{xx} = 6x$, $B = f_{xy} = -6$, $C = f_{yy} = 2$, *so* $AC - B^2 = 12x - 36$.

At $(x, y) = (0, 0)$, $AC - B^2 < 0$ so $(0, 0)$ is a saddle point.
At $(x, y) = (6, 18)$, $AC - B^2 > 0$ and $A > 0$, so $(6, 18)$ is a local minimum.

(15) 4. Let $f(x, y) = x^2 + y^2 - 2y$.

(a) Find the absolute maximum and absolute minimum of f on the ellipse $x^2 + 2y^2 = 8$. (Use Lagrange multipliers.)

Solution: *The objective function is $f(x, y) = x^2+y^2-2y$ and the constraint function is $g(x, y) = x^2 + 2y^2$. The Lagrange multiplier equations are*

$$2x = \lambda \cdot 2x$$
$$2y - 2 = \lambda \cdot 4y$$
$$x^2 + 2y^2 = 8.$$

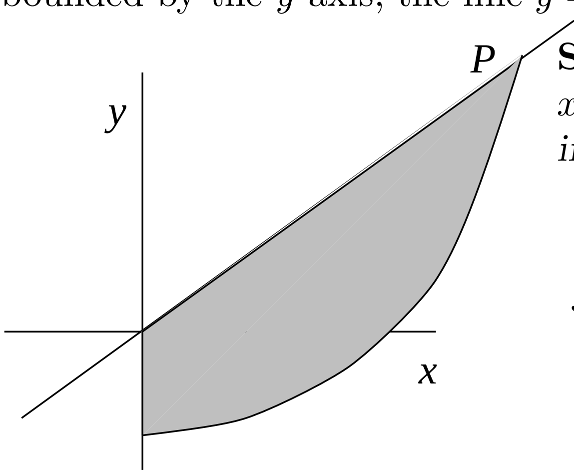
From the first equation, $\lambda = 1$ or $x = 0$. The case $\lambda = 1$ leads to $2y-2 = 4y$, $y = -1$, and then $x = \pm\sqrt{6}$ (from the constraint). The case $x = 0$ leads to $x = \pm 2$ (from the constraint). So the possible max/min points are $(\sqrt{6}, -1)$, $(-\sqrt{6}, -1)$, $(0, 2)$, $(0, -2)$.. The corresponding values of f are 9, 9, 0, 8.. Hence the absolute maximum and minimum occur at $(\pm\sqrt{6}, -1)$ and $(0, 2)$, respectively.

(b) Find the absolute maximum and absolute minimum of f on the "elliptical disk"

$$x^2 + 2y^2 \leq 8.$$

Solution: *The constraint set is closed (because the inequality is $\leq$, not $<$), and bounded, so there is an absolute maximum and absolute minimum. Candidates for these absolute extrema on the boundary are the points found in (a). Candidates in the interior are critical points of f in the interior. These come from $f_x = f_y = 0$, $2x = 2y - 2 = 0$, $(x, y) = (0, 1)$. Since $f(0, 1) = -1$, f has absolute minimum of -1 at $(0, 1)$, and has the same absolute maximum as in (a).*

(10) 5. Compute $\iint_R x\,dA$ where R is the region to the right of the y-axis and bounded by the y-axis, the line $y = 3x$ and the parabola $y = x^2 - 4$.



Solution: *The curves intersect where $3x = x^2 - 4$, $x = -1$ and 4. At P, $x = 4$. The integral equals*

$$\int_{x=0}^{4} \int_{y=x^2-4}^{3x} x\,dy\,dx = \int_{x=0}^{4} x(3x - (x^2 - 4))\,dx$$
$$= \int_{x=0}^{4} -x^3 + 3x^2 + 4x\,dx = 16.$$

(10) 6. Compute $\iiint_S z^2\, dV$ where S is the quarter of the ball $x^2+y^2+z^2 \le 9$ lying above the xy-plane and satisfying $x \ge 0$.

Solution: *In spherical coordinates, the ball is* $0 \le \rho \le 3$. *The portion lying above the xy-plane corresponds to* $0 \le \phi \le \pi/2$. *The portion corresponding to* $x \ge 0$ *corresponds to* $-\pi/2 \le \theta \le \pi/2$. *Since* $z = \rho\cos\phi$, *the integral is*

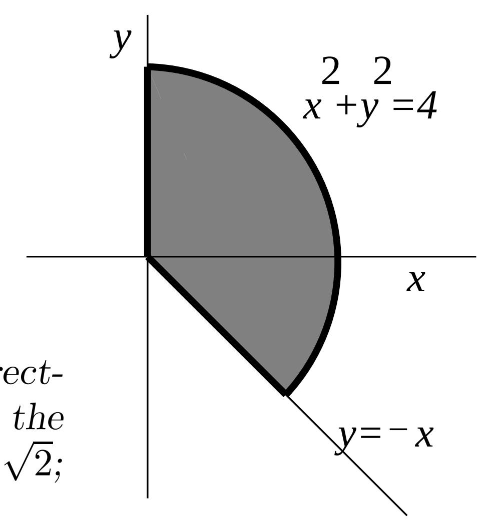
$$\int_{\rho=0}^{3}\int_{\phi=0}^{\pi/2}\int_{\theta=-\pi/2}^{\pi/2} \rho^2\cos^2\phi \cdot \rho^2 \sin\phi\, d\theta\, d\phi\, d\rho = \pi \int_{\rho=0}^{3} \rho^4\, d\rho \int_{\phi=0}^{\pi/2} \cos^2\phi \sin\phi\, d\phi$$
$$= \frac{\pi\cdot 3^5}{5}\cdot\frac{1}{3} = \frac{81\pi}{5},$$

using the rectangle trick and the substitution $u = \cos\phi$.

Note: The integral can also be evaluated in cylindrical coordinates, in which it sets up as $\int_{\theta=-\pi/2}^{\pi/2}\int_{r=0}^{3}\int_{z=0}^{\sqrt{3^2-r^2}} z^2 r\, dz\, dr\, d\theta$.

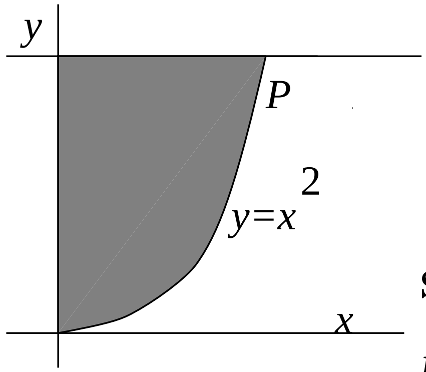
(10) 7. Compute $\iint_R x\, dA$, where R is the region to the right of the y-axis and bounded by the circle of radius 2 centered at the origin, the positive part of the y-axis and the line $y = -x$.

Solution: $\int_{r=0}^{2}\int_{\theta=-\pi/4}^{\pi/2} (r\cos\theta)\cdot r\, dr\, d\theta =$
$\int_{r=0}^{2}\int_{\theta=-\pi/4}^{\pi/2} (r^2\cos\theta)\, dr\, d\theta =$ *(by rect. trick)*
$\int_{r=0}^{2} r^2\, dr \int_{\theta=-\pi/4}^{\pi/2} \cos\theta\, d\theta = \frac{2^3}{3}\cdot\left(1+\frac{\sqrt{2}}{2}\right)$.



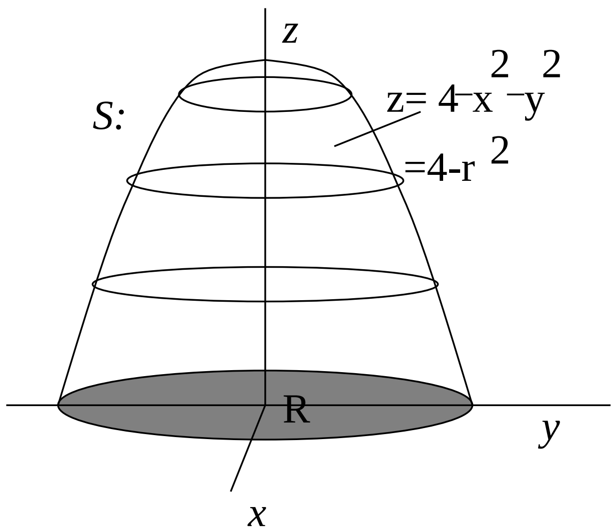
Note. In order to calculate the integral in rectangular coordinates, you must break it up as the sum of two integrals. For Type I break at $x = \sqrt{2}$; *fo Type II break at* $y = 0$.

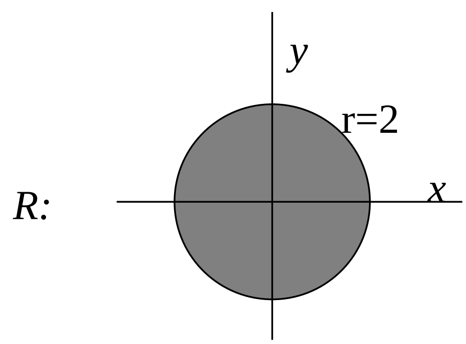
(10) 8. Sketch the region of integration for an iterated integral $\int_{x=0}^{2}\int_{y=x^2}^{4} f(x,y)\, dy\, dx$ and rewrite (but do not attempt to evaluate) the integral with the order of integration interchanged.



Solution: *The point P is $(2,4)$. The reversed integral is* $\int_{y=0}^{4}\int_{x=0}^{\sqrt{y}} f(x,y)\,dx\,dy$

(10) 9. Compute $\iiint\limits_S \sqrt{x^2+y^2}\,dV$ where S is the solid under the paraboloid $z = 4 - x^2 - y^2$ and above the xy-plane.





Solution: *Use cylindrical coordinates because of the rotational symmetry about the z-axis. The top surface of S is the surface $z = 4-x^2-y^2 = 4-r^2$. The bottom surface is $z = 0$. The projection or shadow of S in the xy-plane is a disk whose boundary occurs where the paraboloid $z = 4-r^2$ meets the xy plane, i.e., $4-r^2 = 0$, $r = 2$. This disk R is described by $0 \le r \le 2$, $0 \le \theta \le 2\pi$. Finally the integrand $\sqrt{x^2+y^2}$ equals r and the Jacobian is r. Then*

$$\iiint\limits_S \sqrt{x^2+y^2}\,dV = \iint\limits_R \left[\int_{z=0}^{4-r^2} r\,dz\right] dA$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{2}\int_{z=0}^{4-r^2} r\cdot r\,dz\,dr\,d\theta = \int_{\theta=0}^{2\pi}\int_{r=0}^{2}\int_{z=0}^{4-r^2} r^2\,dz\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{2} r^2 z\Big|_{z=0}^{z=4-r^2} dr\,d\theta = \int_{\theta=0}^{2\pi}\int_{r=0}^{2} r^2(4-r^2)\,dr\,d\theta$$

$$= 2\pi\int_{r=0}^{2} r^2(4-r^2)\,dr = \frac{128\pi}{15}.$$