A The problem

This is #38 in 14.8. Let the dimensions of the box be x, y, z. We want to

$$\begin{array}{rr} \text{maximize or minimize} & V(x,y,z) = xyz \\ \text{subject to the constraints} & g_1(x,y,z) = 2xy + 2xz + 2yz = 1500 \\ \text{and} & g_2(x,y,z) = 4x + 4y + 4z = 200. \end{array}$$

There are also the implicit constraints that x, y, z are positive. Ignore those for the moment. (See D below.)

B The $3 + 2 = 5$ Lagrange multiplier equations are

$$\nabla V = \lambda \nabla g_1 + \mu \nabla g_2 \quad \begin{cases} yz = 2\lambda(y+z) + 4\mu & (1) \\ xz = 2\lambda(x+z) + 4\mu & (2) \\ xy = 2\lambda(x+y) + 4\mu & (3) \end{cases}$$

$$2xy + 2xz + 2yz = 1500 \qquad (4)$$

$$4x + 4y + 4z = 200. \qquad (5)$$

C Finding the critical points

Subtracting (2) from (1): $(y - x)z = 2\lambda(y - x)$. Therefore

$$y = x, \text{ or } z = 2\lambda. \qquad (6)$$

Similarly, using the symmetry of (1), (2) and (3),

$$z = y, \text{ or } x = 2\lambda; \qquad \text{and} \qquad z = x, \text{ or } y = 2\lambda.$$

If $x = y = z$, then (4) and (5) imply $6x^2 = 1500$ and $12x = 200$. But these are inconsistent: $x = \pm\sqrt{250} \neq 200/12$. (A cube cannot satisfy the constraints!) Therefore x, y and z cannot all be equal. In particular, they can't all equal 2λ. By the symmetry of x, y, z, we may assume that $z \neq 2\lambda$. Therefore $y = x$ by (6).

Now (4) and (5) give

$$2x^2 + 4xz = 1500, \qquad 8x + 4z = 200.$$

Therefore $2x^2 + x(200 - 8x) = 1500$, so $x = (50 \pm 5\sqrt{10})/3$, $y = x$, and $z = 50 - 2x = (50 \mp 10\sqrt{10})/3$. The critical points are

$$P\left(\frac{50+5\sqrt{10}}{3}, \frac{50+5\sqrt{10}}{3}, \frac{50-10\sqrt{10}}{3}\right) \qquad \text{and}$$

$$Q\left(\frac{50-5\sqrt{10}}{3}, \frac{50-5\sqrt{10}}{3}, \frac{50+10\sqrt{10}}{3}\right),$$

and the four other points obtained by permuting x, y, z. We can ignore the permutations, since they don't give any new ways to design the box.

D Do maxima and minima necessarily exist? Analyzing the constraint set

The constraint set is defined by the constraints (4) and (5) and the implicit constraints

$$x > 0,\ y > 0,\ z > 0. \qquad (7)$$

By these and (5), $x \leq 50$, $y \leq 50$, $z \leq 50$. Therefore the constraint set is **bounded**. It is not obvious that the constraint set is closed, because the inequalities in (7) are strict. However, the constraints (4) and (5) **prohibit** x, y and z from equalling 0. For instance if $x = 0$, then by (4), $yz = 750$ and by (5), $y + z = 50$. Therefore $y + (750/y) = 50$, $y^2 - 50y + 750 = 0$, which has no real solutions. Similarly, y and z cannot be 0. Therefore the constraints (7) have the same effect as

$$x \geq 0,\ y \geq 0,\ z \geq 0. \qquad (8)$$

So the constraint set is defined by the weak inequalities (4), (5) and (8), therefore is closed. Since $V = xyz$ is continuous in x, y, z, and since the constraint set is closed and bounded, there exist maximum and minimum points, by the Extreme Value Theorem.

E The solution

Therefore whichever of P and Q gives the largest (resp. smallest) volume is the max (resp. min). Let $\alpha^{\pm} = (50 \pm 5\sqrt{10})/3$ and $\beta^{\pm} = (50 \pm 10\sqrt{10})/3$. That is $\alpha^{+} = (50 + 5\sqrt{10})/3 \approx 22$, $\alpha^{-} = (50 - 5\sqrt{10})/3 \approx 11.3$, etc.

At P, the volume is $V = (\alpha^{+})^2\beta^{-} = 2500(35 - \sqrt{10})/27 \approx 3000$. This is the **min**, and the box is $\alpha^{+} \times \alpha^{+} \times \beta^{-}$ which is approximately $22 \times 22 \times 6$.

At Q, the volume is $V = (\alpha^{-})^2\beta^{+} = 2500(35 + \sqrt{10})/27 \approx 3500$. This is the **max**, and the box is $\alpha^{-} \times \alpha^{-} \times \beta^{+}$ which is approximately $11.3 \times 11.3 \times 27.3$.