SOLUTIONS TO REVIEW PROBLEMS FOR EXAM #2, 251-04-05-06, FALL 2005

1. Find equations for the tangent plane and normal line to the surface $x^4y+y^4z+2z^4x=4$ at the point $(1,1,1)$.

SOLUTION:

The surface is a level surface of $F(x,y,z)=x^4y+y^4z+2z^4x$. Therefore a normal vector is

$$\vec{n}=\vec{\nabla}(x^4y+y^4z+2z^4x)\Big|_{(1,1,1)}=\left\langle 4x^3y+2z^4, x^4+4y^3z, y^4+8z^3x\right\rangle\Big|_{(1,1,1)}=\langle 6,5,9\rangle.$$

The tangent plane is $6(x-1)+5(y-1)+9(z-1)=0$, and parametric equations for the normal line are

$$x=1+6t,\ y=1+5t,\ z=1+9t.$$

2. The equation $x^3y\cos z+2x^2z-5e^{yz}=3$ defines z as a function of x and y near the point $(2,1,0)$. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ as functions of x, y, and z, and evaluate them at $(2,1,0)$.

SOLUTION:

Regard $z=z(x,y)$ as a function of x and y. Then both sides of the given equation are functions of x and y. Take $\dfrac{\partial}{\partial x}$ of both sides and solve for $\dfrac{\partial z}{\partial x}$:

$$3x^2y\cos z+x^3y(-\sin z\frac{\partial z}{\partial x})+4xz+2x^2\frac{\partial z}{\partial x}-5e^{yz}\cdot y\frac{\partial z}{\partial x}=0,$$

$$\frac{\partial z}{\partial x}=-\frac{3x^2y\cos z+4xz}{-x^3y\sin z+2x^2-5ye^{yz}}.$$

Similarly, take $\dfrac{\partial}{\partial y}$ of both sides of the original equation:

$$x^3\cos z-x^3y\sin z\frac{\partial z}{\partial y}-5e^{yz}(z+y\frac{\partial z}{\partial y})=0$$

$$\frac{\partial z}{\partial y}=-\frac{x^3\cos z-5ze^{yz}}{-x^3y\sin z+2x^2-5ye^{yz}}.$$

Alternative method (the same method, really, but by formula): The relationship among x,y,z is $F(x,y,z)$ =constant, where $F(x,y,z)=x^3y\cos z+2x^2z-5e^{yz}$. In such a situation the following general formulas apply:

$$\frac{\partial z}{\partial x}=\frac{-\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}}\qquad\text{and}\qquad\frac{\partial z}{\partial y}=\frac{-\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}}$$

Either way, at $(x,y,z)=(2,1,0)$, $\dfrac{\partial z}{\partial x}=-12/3$ and $\dfrac{\partial z}{\partial y}=-8/3$.

3. Find the absolute maximum and minimum values of the function $f(x, y, z) = x - 2y + 3z$ on the ellipsoid $x^2 + 2y^2 + 3z^2 = 24$.

SOLUTION:

The ellipsoid is closed (it is a surface in $\boldsymbol{R}^3$) and bounded (no point on it has a coordinate bigger than 10, say). Also f is continuous (everywhere). So there exist absolute max and min points by the Extreme Value Theorem. The Lagrange multiplier equations are

$$\begin{aligned} 1 &= \lambda \cdot 2x \\ -2 &= \lambda \cdot 4y \\ 3 &= \lambda \cdot 6z \\ x^2 + 2y^2 + 3z^2 &= 24 \end{aligned}$$

the first three being $\vec{\nabla} f = \lambda \vec{\nabla} g$, where the constraint has the form $g(x)$ =constant. Therefore $x = 1/2\lambda$, $y = -1/2\lambda$, $z = 1/2\lambda$. We can substitute in the constraint, solve for λ and then determine x, y, z. Or, observe that these equations imply $x = -y = z$. Substitute into the constraint to get $x^2 + 2x^2 + 3x^2 = 24$, $x = \pm 2$ (and then $y = -x = \mp 2$ and $z = x = \pm 2$). The two max/min candidates are $(2, -2, 2)$ and $(-2, 2, -2)$. Since existence of max/min points is assured, $f(2, -2, 2) = 12$ and $f(-2, 2, -2) = -12$ are the max and min values.

4. Find all the critical points of $f(x, y) = x^3 - 6xy + y^3$, and classify them as local maxima, local minima, or saddle points.

SOLUTION:

For critical points, solve

$$\begin{aligned} \frac{\partial f}{\partial x} &= 3x^2 - 6y = 0 \\ \frac{\partial f}{\partial y} &= -6x + 3y^2 = 0 \end{aligned}$$

giving $x^2 = 2y$ and $y^2 = 2x$, so $x^4 = 4y^2 = 8x$, $x(x^3 - 8) = 0$, $x = 0$ or $x = 2$.

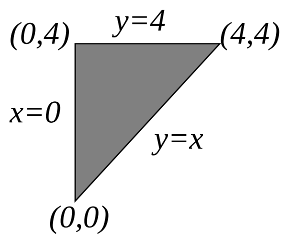
Then $y = x^2/2 = 0$ or 2, respectively. The critical points are $(0, 0)$ and $(2, 2)$.

Next, $A = \dfrac{\partial^2 f}{\partial x^2} = 6x$, $B = \dfrac{\partial^2 f}{\partial x \partial y} = -6$ and $C = \dfrac{\partial^2 f}{\partial y^2} = 6y$.

crit.pt.	A	B	C	$AC - B^2$	type
$(0, 0)$	0	-6	0	-36	saddle
$(2, 2)$	12	-6	12	108	local min

5. For the function $f(x, y) = x^2 - xy + y^2 - 3y$, find the absolute maximum and minimum values on the triangular region bounded by the lines $x = 0$, $y = 4$, and $y = x$.

SOLUTION: The triangular region is closed and bounded and the function f is continuous, so absolute max and min exist. They can only occur (a) in the interior at a critical point of f and (b) on the boundary. The boundary has to be considered one side at a time.



Let $z = f(x,y)$. For the critical point(s), solve

$\dfrac{\partial z}{\partial x} = 2x - y = 0$ and $\dfrac{\partial z}{\partial y} = -x + 2y - 3 = 0$, giving $y = 2x$, $3x - 3 = 0$, $(x, y) =, 2)A(\mathbf{1}, \mathbf{2})$.

On the side $x = 0$, $0 \le y \le 4$, $z = y^2 - 3y$. The possible max/min points are at $\dfrac{dz}{dy} = 2y - 3 = 0$, $(x, y) = (0, 1.5)$. The endpoints have also to be considered.

On the side $y = 4$, $0 \le x \le 4$, $z = x^2 - 4x + 4$. Then at $\dfrac{dz}{dx} = 0$, $(x, y) = (2, 4)$, and the endpoints as well are possibilities.

On the side $y = x$, $0 \le x \le 4$, $z = x^2 - x^2 + x^2 - 3x = x^2 - 3x$, and $\dfrac{dz}{dx} = 2x - 3 = 0$. So $(x, y) = (1.5, 1.5)$ and the endpoints must be considered.

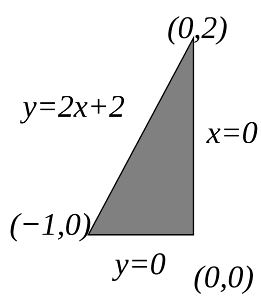
The candidates are $(x, y) = (1, 2)$, $(0, 1.5)$, $(2, 4)$, $(1.5, 1.5)$, and the three vertices, $(0, 0)$, $(0, 4)$, $(4, 4)$. Correspondingly $z = -3$, -2.25, 0, -2.25, 0, 4, 4. The max is $z = 4$, at $(0, 4)$ and $(4, 4)$; the min is $z = -3$, at $(1, 2)$.

6. Compute $\displaystyle\iint_R 12x\, dA$ where R is the triangle in the second quadrant $(x \le 0, y \ge 0)$ enclosed by the coordinate axes and the line $2x - y + 2 = 0$.

SOLUTION: There is a choice: the region R is both type I $(-1 \le x \le 0,\ 0 \le y \le 2x + 2)$ and type II $(0 \le y \le 2,\ (y - 2)/2 \le x \le 0)$. Working the integral as type I we get

$$\int_{x=-1}^{0} \int_{y=0}^{2x+2} 12x\, dA = \int_{x=-1}^{0} 12xy\Big|_{y=0}^{y=2x+2} dx$$

$$= \int_{x=-1}^{0} 12x(2x + 2)\, dx = -4$$



7. Sketch the region of integration for the integral $\displaystyle\int_0^1 \int_{3x^2}^{3x} 2xy\, dy\, dx$. Write an equivalent integral with the order of integration reversed. Evaluate both integrals and check that they are equal.

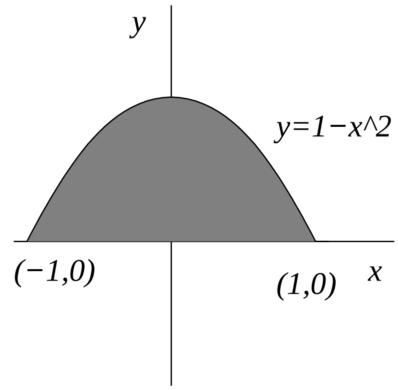
SOLUTION:

$$\int_{y=0}^{3}\int_{x=y/3}^{\sqrt{y/3}} 2xy\,dx\,dy = \int_{y=0}^{3} x^2y\Big|_{x=y/3}^{x=\sqrt{y/3}}\,dy = \int_0^3 y[(y/3 - y^2/9)]\,dy = \frac{y^3}{9} - \frac{y^4}{36}\Big|_0^3 = 3/4$$

$$\int_{x=0}^{1}\int_{y=3x^2}^{3} 2xy\,dy\,dx = \int_{x=0}^{1} xy^2\Big|_{y=3x}^{y=3x^2}\,dx = \int_0^1 9x^3 - 9x^5\,dx = 3/4$$

8. Find the center of mass of a uniform plane lamina whose boundary is formed by the x-axis and the parabola $y = 1 - x^2$.

SOLUTION: The density is uniform, i.e. a constant c. Since the lamina and its density are symmetric with respect to the y-axis, $\overline{x} = 0$.



$$\overline{y} = \frac{M_x}{M} \quad \text{with}$$

$$M_x = \iint_D y\,dA = \int_{x=-1}^{1}\int_{y=0}^{1-x^2} cy\,dy\,dx$$

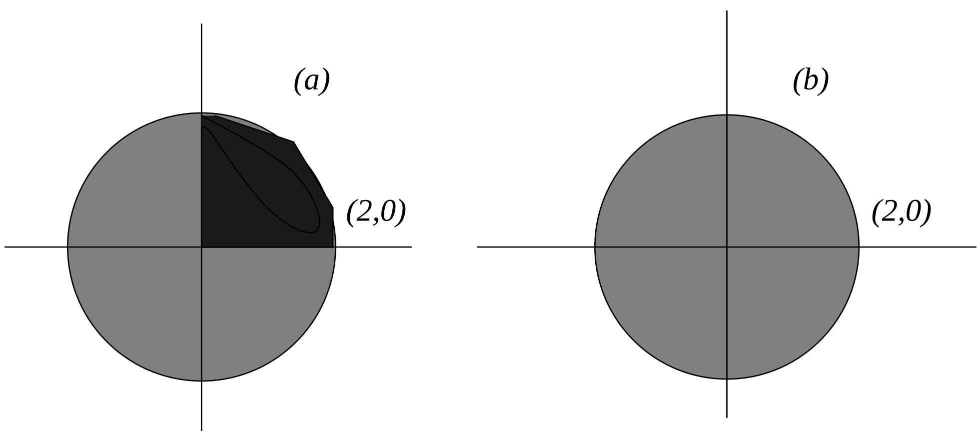
$$= c\int_{x=-1}^{1} \frac{y^2}{2}\Big|_{y=0}^{y=1-x^2} = c\int_{x=-1}^{1} \frac{(1-x^2)^2}{2}\,dx = \frac{8c}{15}$$

and $M = \iint_D c\,dA = c\int_{x=-1}^{1}\int_{y=0}^{1-x^2} dy\,dx = c\int_{x=-1}^{1}(1-x^2)\,dx = \frac{4c}{3}$.

Therefore $\overline{y} = \dfrac{8c/15}{4c/3} = \dfrac{2}{5}$ so $(\overline{x},\overline{y}) = \left(0, \dfrac{2}{5}\right)$.

9. Use polar coordinates to find: (a) $\displaystyle\int_{x=0}^{2}\int_{y=0}^{\sqrt{4-x^2}} xy^2\,dy\,dx$ and (b) $\displaystyle\int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} e^{-2x^2-2y^2}\,dy\,dx$.

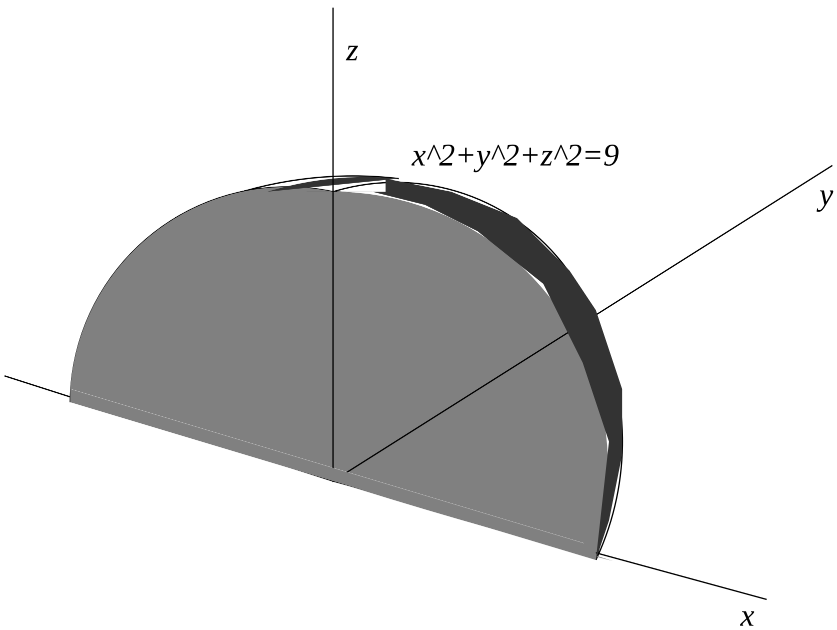
SOLUTION:



(a) $\int_{r=0}^{2}\int_{\theta=0}^{\pi/2}(r\cos\theta)(r\sin\theta)^2 r\,dr\,d\theta = \int_{r=0}^{2} r^4\,dr\int_{\theta=0}^{\pi/2}\cos\theta\sin^2\theta\,d\theta = \frac{32}{5}\cdot\frac{1}{3} = \frac{32}{15}$. (substitute $u = \sin\theta$).

(b) $\int_{r=0}^{2}\int_{\theta=0}^{2\pi} e^{-2r^2} r\,dr\,d\theta = 2\pi\int_{r=0}^{2} e^{-2r^2} r\,dr = 2\pi\int_{u=0}^{-8}(-1/4)\,du = \frac{\pi}{2}(1-e^{-8})$.

10. Consider the integral $\iiint\limits_S (x^2+y^2)\,dV$ where S is that quarter of the ball $x^2+y^2+z^2\le 9$ for which $y\ge 0$ and $z\ge 0$. Set up the integral in two ways, first using cylindrical coordinates and then using spherical coordinates. Evaluate both ways and check that the results are the same.



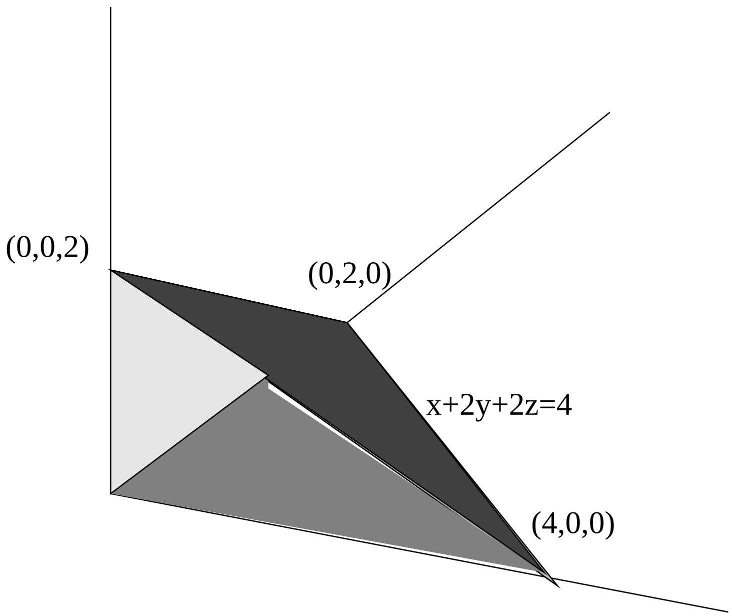
SOLUTION: First, in cylindrical coordinates, S is described by $0\le z\le\sqrt{9-r^2}$, over the semi-disk $0\le r\le 3$, $-\pi/2\le\theta\le\pi/2$ in the x,y-plane. Or equivalently for each z, $0\le r\le sqrt9-z^2$, with $0\le z\le 3$ and $-\pi/2\le\theta\le\pi/2$.
In spherical coordinates, S is described by $0\le\rho\le 3$, $-\pi/2\le\theta\le\pi/2$, $0\le\phi\le\pi/2$.
Also $x^2+y^2=\rho^2\sin^2\phi(\cos^2\theta+\sin^2\theta)=\rho^2\sin^2\phi$.

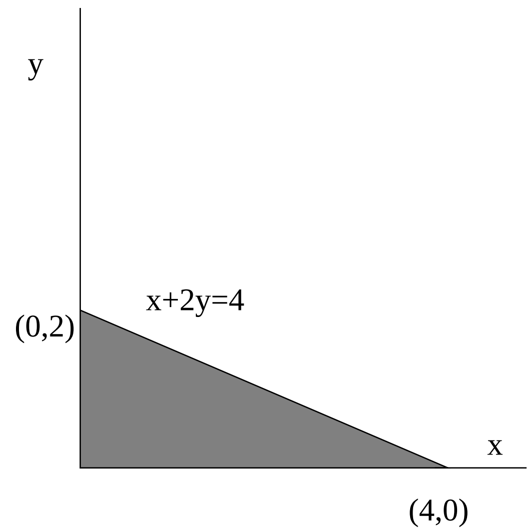
$$\int_{\theta=-\pi/2}^{\pi/2}\int_{z=0}^{3}\int_{r=0}^{\sqrt{9-z^2}} r^2\cdot r\,dr\,dz\,d\theta = \pi\int_{z=0}^{3}\left.\frac{r^4}{4}\right|_0^{\sqrt{9-z^2}}dz$$
$$=\frac{\pi}{4}\int_{z=0}^{3}(9-z^2)^2\,dz = \frac{162\pi}{5}$$
$$\int_{\theta=-\pi/2}^{\pi/2}\int_{\phi=0}^{\pi/2}\int_{\rho=0}^{3}\rho^2\sin^2\phi\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_{-\pi/2}^{\pi/2}d\theta\int_0^{\pi/2}\sin^3\phi\,d\phi\int_0^3\rho^4\,d\rho = \pi\cdot\frac{2}{3}\cdot\frac{3^5}{5}$$
$$=\frac{162\pi}{5}$$

(Note: with cylindrical coordinates it is also correct to have limits of integration as follows, reading from left to right: $-\pi/2\le\theta\le\pi/2$, $0\le r\le 3$, $0\le z\le\sqrt{9-r^2}$. However, this leads to the intermediate integral $\int r^3\sqrt{9-r^2}$, still do-able ($u = 9-r^2$, $r^3 = r^2 \cdot r = (9-u)(-0.5)\,du$) but harder than the order of integration above.)

11. Evaluate $\displaystyle\iiint_S y\,dV$ where S is the solid in the first octant cut off by the plane $x + 2y + 2z = 4$.

SOLUTION:





Using the z-axis as the "up" axis, the floor is the x,y-plane, the ceiling is $x + 2y + 2z = 4$, and the shadow in the x,y-plane is the triangle $0 \le x \le 4$, $0 \le y \le (4-x)/2$.

$$\begin{aligned}
y\,dV &= \int_{x=0}^{4}\int_{y=0}^{(4-x)/2}\int_{z=0}^{(4-x-2y)/2} y\,dz\,dy\,dx = \int_{x=0}^{4}\int_{y=0}^{(4-x)/2} yz\Big|_{z=0}^{z=(4-x-2y)/2} \\
&= \int_{x=0}^{4}\int_{y=0}^{(4-x)/2} y(4-x-2y)/2\,dy\,dx = \int_{x=0}^{4}\left(\frac{(4-x)y^2}{4} - \frac{y^3}{3}\right)\Big|_{y=0}^{y=(4-x)/2} dx \\
&= \int_0^4\left(\frac{(4-x)^3}{16} - \frac{(4-x)^3}{24}\right) dx = \frac{4}{3}
\end{aligned}$$

Alternative computation: The fractions can be finessed by making the substitution $u = x/4$, $v = 2y/4 = y/2$, $w = 2z/4 = z/2$. The new solid in u,v,w-space is enclosed by the coordinate planes and the plane $u + v + w = 1$. The integrand is $y = 2v$. The Jacobian is

$$J = \left|\;\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\;\right| = \left|\;\det\begin{vmatrix}4 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{vmatrix}\;\right| = 16, \text{ and}$$

$$\begin{aligned}
\iiint_S y\,dA &= \int_{u=0}^{1}\int_{v=0}^{1-u}\int_{w=0}^{1-u-v} 2v\cdot 16\,dw\,dv\,du = 16\int_{u=0}^{1}\int_{v=0}^{1-u} 2v(1-u-v)\,dv\,du \\
&= 16\int_{u=0}^{1}(1-u)v^2 - \frac{2v^3}{3}\Big|_{v=0}^{v=1-u} = 16\int_0^1\frac{(1-u)^3}{3}\,du = \frac{16}{12} = \frac{4}{3}.
\end{aligned}$$

12. Compute $\int_C xy\,dx + y^2\,dy$ where C is the quarter-circle from $(2,0)$ to $(0,2)$ (centered at the origin).

SOLUTION:

Any parametrization will do. If you use $x = 2\cos t$, $y = 2\sin t$, $0 \le t \le \pi/2$, then $dx = -2\sin t\,dt$, $dy = 2\cos t\,dt$ and

$$\int_C xy\,dx + y^2\,dy = \int_{t=0}^{\pi/2} \left[2\cos t \cdot 2\sin t \cdot (-2\sin t) + (2\sin t)^2 \cdot 2\cos t\right] dt = \int_{t=0}^{\pi/2} 0\,dt = 0.$$

13. Compute $\int_C xy\,dx + y^2\,dy$ where C is the straight line segment from $(2,0)$ to $(0,2)$.

SOLUTION:

The line is $x + y = 2$. Either x or y may be used as the parameter. Using y as the parameter: $x = 2 - t$, $y = t$, $0 \le t \le 2$. Then $dx = -\,dt$ and $dy = dt$, so

$$\int_C xy\,dx + y^2\,dy = \int_{t=0}^{2} \left[(2-t)\cdot t \cdot (-1) + t^2\right] dt = \int_0^2 (2t^2 - 2t)\,dt = \frac{4}{3}.$$

14. Compute $\int_C (x + 2y)\,ds$ where C is the straight line segment from $(2,0)$ to $(0,2)$.

SOLUTION:

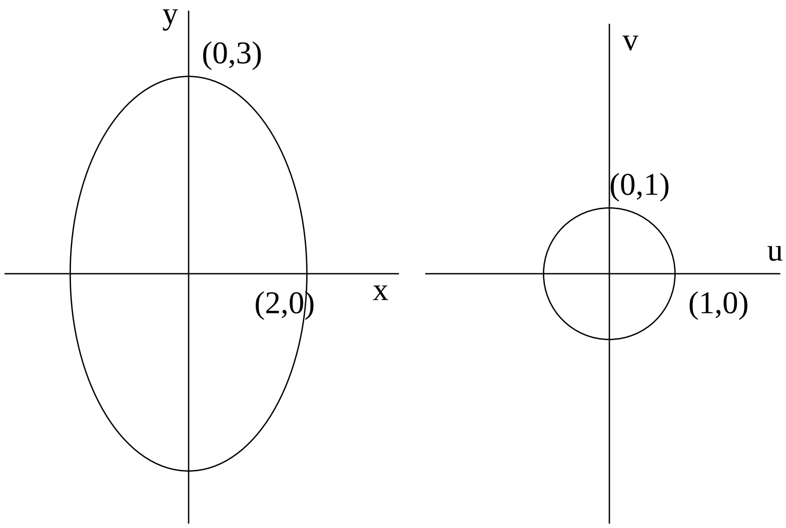
Again use $x = 2 - t$, $y = t$, $0 \le t \le 2$. Then

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \sqrt{2}\,dt$$

$$\int_C (x + 2y)\,dx = \int_0^2 (2-t) + 2t \cdot \sqrt{2}\,dt = \sqrt{2}\int_0^2 (t+2)\,dt = 6\sqrt{2}.$$

15. Find the Jacobian of the transformation $x = 2u$, $y = 3v$. Use this transformation as a first step to compute $\displaystyle\iint_R x^2\,dA$ where R is the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} \le 1$.

SOLUTION:



R S

$$J = \left| \det \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right| = \left| \det \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} \right| = 6.$$

The elliptical disk R is transformed to the unit circular disk S $(u^2 + v^2 \leq 1)$ in the u, v-plane, since $u = x/2$ and $y = v/3$. Then using polar coordinates in the u, v-plane, $u = r\cos\theta$, $v = r\sin\theta$,

$$\iint\limits_R x^2\, dA = \iint\limits_S (2u)^2 \cdot 6dA' = \int_{r=0}^{1} \int_{\theta=0}^{2\pi} (2r\cos\theta)^2 \cdot 6 \cdot r\, dr\, d\theta = 24 \int_0^1 r^3\, dr \cdot \int_0^{2\pi} \cos^2\theta\, d\theta = 6\pi.$$