REVIEW PROBLEMS ON CHAPTER 16, 251-04-05-06, FALL 2005

The exam will cover the entire syllabus. You will be allowed to use a calculator without symbolic capabilities, such as TI-82, 83, 85, 86. You may not bring a crib sheet. You will be given the formulas for Stokes' Theorem and the Divergence Theorem.

The following problems are mostly taken or adapted from recent 251 exams. (Answers are not guaranteed.)

1. Let C be the triangle in the plane with vertices $(0,1)$, $(1,0)$ and $(1,1)$, oriented counter-clockwise.

(a) Evaluate $\oint_C 2y^2\,dx + 2x\,dy$ directly as a line integral.

(b) Evaluate the same line integral by using Green's Theorem. Ans. $-1/3$

2. Consider the vector field

$$\boldsymbol{F}(x,y) = (2x(y^3-1) + 2y\ln y)\,\boldsymbol{i} + (2x\ln y + 2(x+y) + c(xy)^2)\,\boldsymbol{j}$$

where c is a constant.

(a) What is the domain of $\boldsymbol{F}$? Is it simply connected?

(b) Determine the value(s) of c for which $\boldsymbol{F}$ is conservative. Then find a function f such that $\nabla f = \boldsymbol{F}$ everywhere in the domain of $\boldsymbol{F}$.

(c) **Quickly** calculate $\int_C \boldsymbol{F} \bullet d\boldsymbol{r}$ where C is the path $\boldsymbol{r}(t) = \cos\pi t\,\boldsymbol{i} + (e^{2t^4})\,\boldsymbol{j}$, $0 \le t \le 1$.

Ans. $c = 3$, $y > 0$, yes, $x^2(y^3-1) + 2xy\ln y + y^2 + C$, $e^3 + e^2 - 2e - 2$.

3. Let E be the solid region above the plane $z = 3$ and below the paraboloid $z = 4 - x^2 - y^2$. Let Σ be the boundary surface of E. Let $\boldsymbol{F}(x,y,z) = (y+x)\,\boldsymbol{i} + (y-x)\,\boldsymbol{j}$. Verify the Divergence Theorem

$$\iiint_E \operatorname{div}\boldsymbol{F}\,dV = \iint_\Sigma \boldsymbol{F} \bullet \boldsymbol{n}\,dS \qquad (\boldsymbol{n} = \text{outward normal})$$

by computing both sides separately and showing that they are equal. Ans. $\pi = \pi$

4. Use the Divergence Theorem to calculate $\iint_\Sigma \boldsymbol{F} \bullet \boldsymbol{n}\,dS$ where Σ is the (entire) surface of the cylinder enclosed by the surfaces $x^2 + y^2 = 1$, $z = 1$ and $z = 3$, and $\boldsymbol{F} = (x^3 + 2y)\,\boldsymbol{i} + (2x + y^3)\,\boldsymbol{j} + 3z\,\boldsymbol{k}$. Ans. 9π

5. Find the surface area of that portion of the surface $z = x^2 + y^2 + 3$ lying between the planes $z = 5$ and $z = 9$. Ans. $49\pi/3$

6. Use Green's Theorem to calculate the work done by the force field $\boldsymbol{F} = (x - y^2)\,\boldsymbol{i} + (x^2 - y)\,\boldsymbol{j}$ on a particle moving in straight line segments from $(2,1)$ to $(2,4)$ to $(3,4)$ to $(3,1)$ and back to $(2,1)$. Ans. -30

7. Let $\boldsymbol{F}(x,y,z) = (x+y)\,\boldsymbol{i} + (y-x)\,\boldsymbol{j} + z\,\boldsymbol{k}$.

(a) Compute $\oint_C \boldsymbol{F} \bullet d\boldsymbol{r}$ where C is the circle $x^2 + y^2 = 4$ in the xy-plane, oriented clockwise.

(b) Compute $\iint_\Sigma \operatorname{curl}\boldsymbol{F} \bullet \boldsymbol{n}\,dS$ where Σ is the upper half of the sphere $x^2 + y^2 + z^2 = 4$, and $\boldsymbol{n}$ points up.

(c) Reconcile your answers to (a) and (b) with Stokes' Theorem. Ans. (a) 8π (b) -8π.

8. Let Σ_1 be the disk $x^2 + y^2 \le 1$ in the xy-plane, oriented with $\boldsymbol{n}$ pointing up. Let Σ_2 be the "northern hemisphere" $x^2+y^2+z^2 = 1$, $z \ge 0$, also oriented with $\boldsymbol{n}$ pointing up. Let $\boldsymbol{F}$ be any twice differentiable vector field. Give two explanations why

$$\iint_{\Sigma_1} \operatorname{curl}\boldsymbol{F} \bullet \boldsymbol{n}\,dS = \iint_{\Sigma_2} \operatorname{curl}\boldsymbol{F} \bullet \boldsymbol{n}\,dS$$

(a) one using Stokes' Theorem

(b) the other using the Divergence Theorem (first show that $\operatorname{div}\operatorname{curl}\boldsymbol{F} = 0$).