Some solutions to last assignment

16.5/#4

$$\begin{aligned}\nabla \times \mathbf{F} &= \det\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & \cos xz & -\sin xy \end{vmatrix} \\ &= (-x\cos xy + x\sin xz)\,\mathbf{i} + y\cos xy\,\mathbf{j} - z\sin xz\,\mathbf{k} \\ \nabla \bullet \mathbf{F} &= 0.\end{aligned}$$

16.5/#12
(i) is a vector field and (l) is a scalar field. (h), (j), (k) are not meaningful. (h): f is not a vector field. (j): div $\mathbf{F}$ is not a vector field. (k): div $\mathbf{F}$ is not a vector field.

16.6/#21 Use spherical coords ϕ, θ as parameters on this spherical surface of radius 2.

$$x = 2\sin\phi\cos\theta, \quad y = 2\sin\phi\sin\theta, \quad z = 2\cos\phi, \qquad 0 \le \phi \le \pi/4,\ 0 \le \theta \le 2\pi$$

$$\mathbf{r} = 2\sin\phi\cos\theta\mathbf{i} + 2\sin\phi\sin\theta\mathbf{j} + 2\cos\phi\mathbf{k}$$

$$dS = |\mathbf{r}_\phi \times \mathbf{r}_\theta|\,d\phi\,d\theta = \left|\det\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos\phi\cos\theta & -2\cos\phi\sin\theta & -2\sin\phi \\ -2\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0 \end{vmatrix}\right| d\phi\,d\theta$$

$$= \ldots = |4\sin^2\phi\cos\theta\mathbf{i} + 4\sin^2\phi\sin\theta\mathbf{j} + 4\cos\phi\sin\phi\mathbf{k}|\,d\phi\,d\theta = 4\sin\phi\,d\phi\,d\theta, \qquad \text{so}$$

$$\text{area} = \int_{\phi=0}^{\pi/4}\int_{\theta=0}^{2\pi} 4\sin\phi\,d\phi\,d\theta = \pi(8 - 4\sqrt{2}).$$

16.7/#5
The surface is part of the graph of the function $z = 1 + 2x + 3y$; use the independent variables x, y as parameters.

$$x = x \qquad y = y \qquad z = 1 + 2x + 3y \qquad 0 \le x \le 3,\ 0 \le y \le 2$$

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + (1 + 2x + 3y)\mathbf{k}$$

$$dS = |\mathbf{r}_x \times \mathbf{r}_y|\,dx\,dy = \left|\det\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix}\right| dx\,dy = |-2\mathbf{i} - 3\mathbf{j} + \mathbf{k}|\,dx\,dy = \sqrt{14}\,dx\,dy$$

$$\iint_S x^2yz\,dS = \int_{x=0}^{3}\int_{y=0}^{2} x^2y(1 + 2x + 3y)^2\sqrt{14}\,dy\,dx = \ldots = 171\sqrt{14}$$

16.7/#19

Surface is part of $z = 4 - x^2 - y^2$; use parameters x, y.

$$
\begin{aligned}
&x = x \qquad y = y \qquad z = 4 - x^2 - y^2 \qquad 0 \le x \le 1,\ 0 \le y \le 1 \\
&\mathbf{r} = x\mathbf{i} + y\mathbf{j} + (4 - x^2 - y^2)\mathbf{k} \\
\mathbf{F} \bullet \mathbf{n}\, dS &= \det \begin{vmatrix} \mathbf{F} \\ \mathbf{r}_x \\ \mathbf{r}_y \end{vmatrix} dx\, dy = \det \begin{vmatrix} xy & y(4 - x^2 - y^2) & (4 - x^2 - y^2)x \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} dx\, dy \\
&= (2x^2y + 2y^2(4 - x^2 - y^2) + x(4 - x^2 - y^2))\, dx\, dy \\
\iint_S \mathbf{F} \bullet \mathbf{n}\, dS &= \int_{x=0}^{1} \int_{y=0}^{1} (2x^2y + 2y^2(4 - x^2 - y^2) + x(4 - x^2 - y^2))\, dx\, dy = \frac{713}{180}
\end{aligned}
$$

16.8/#9

For Stokes' Theorem use a surface Σ bounded by the given C, and integrate $\operatorname{curl}\mathbf{F}$ over Σ. First, compute

$$\operatorname{curl}\mathbf{F} = \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & 2xz & e^{xy} \end{vmatrix} = (xe^{xy} - 2x)\mathbf{i} + (y - ye^{xy})\mathbf{j} + z\mathbf{k}.$$

Here use the disk $x^2 + y^2 \le 16$, $z = 5$, as Σ. Since C is oriented counterclockwise, we must take $\mathbf{n}$ pointing up, by the right hand rule. (In fact $\mathbf{n} = \mathbf{k}$.) For parameters we can use x, y themselves; the surface is part of the graph of the function $z = 5$. Let D be the disk of radius 4 in the x, y plane centered at the origin.

$$x = x, \qquad y = y, \qquad z = 5, \qquad (x, y) \in D$$

$$\operatorname{curl}\mathbf{F} \bullet \mathbf{n}\, dS = \det \begin{vmatrix} xe^{xy} - 2x & y - ye^{xy} & 5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} dx\, dy = 5\, dx\, dy$$

(More easily: $dS = dx\, dy$ since $z = 5$ is parallel to x, y plane;
$\operatorname{curl}\mathbf{F} \bullet \mathbf{n}\, dS = \operatorname{curl}\mathbf{F} \bullet \mathbf{k}\, dS = z\, dx\, dy = 5\, dx\, dy$)

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \iint_\Sigma \operatorname{curl}\mathbf{F} \bullet \mathbf{n}\, dS = \iint_D 5\, dx\, dy = 5 \cdot (\text{area of } D) = 80\pi.$$

16.9/#15

Let A be the solid sphere with center at the origin and radius R. Let $\mathbf{n}$ be the outward normal on the surface of the sphere.

$$\operatorname{div}\mathbf{F} = 12x^2z + 12y^2z + 12z^3 = 12z(x^2+y^2+z^2),$$

$$\iint_S \mathbf{F}\bullet\mathbf{n}\,dS = \iiint_A \operatorname{div}\mathbf{F}\,dV = \iiint_A 12z(x^2+y^2+z^2)\,dV$$

$$\text{(use sph. coord., } J = \rho^2\sin\phi,\ x^2+y^2+z^2=\rho^2,\ z=\rho\cos\phi)$$

$$\iint_S \mathbf{F}\bullet\mathbf{n}\,dS = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{\rho=0}^{R} 12(\rho\cos\phi)(\rho^2)(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$$

$$= 12\cdot 2\pi\int_{\phi=0}^{\pi}\sin\phi\cos\phi\,d\phi\cdot\int_{\rho=0}^{R}\rho^5\,d\rho = 12\cdot 2\pi\cdot\frac{1}{2}\cdot\frac{R^6}{6} = 2\pi R^6.$$

16.9/#19

The idea is that it's probably easier to evaluate $\iint_{S_1}\mathbf{F}\bullet\mathbf{n}\,dS$ than $\iint_S \mathbf{F}\bullet\mathbf{n}\,dS$, and they are related by the divergence theorem:

$$\iint_{S_1}\mathbf{F}\bullet\mathbf{n}\,dS + \iint_S \mathbf{F}\bullet\mathbf{n}\,dS = \iint_{S_2}\mathbf{F}\bullet\mathbf{n}\,dS = \iiint_A \operatorname{div}\mathbf{F}\,dV.$$

Here $\mathbf{n}$ points outward on S_2, so it points **downward** on S_1. Thus $\mathbf{n} = -\mathbf{k}$ on S_1. Also A is the solid hemisphere bounded by S and S_1: in spherical coordinates, $0\le\rho\le 1$, $0\le\phi\le\pi/2$, $0\le\theta\le 2\pi$.
First, for S_1, $\mathbf{n} = -\mathbf{k}$ and since S_1 is part of the x,y plane, $dS = dx\,dy$, the parameters are x and y, and they vary over the unit disk D in the x,y-plane. Also $z=0$ on S_1 so

$$\mathbf{F}\bullet\mathbf{n}\,dS = \mathbf{F}\bullet(-\mathbf{k})\,dx\,dy = -(x^2z+y^2)\,dx\,dy = -y^2\,dx\,dy$$

$$\iint_{S_1}\mathbf{F}\bullet\mathbf{n}\,dS = -\iint_D y^2\,dx\,dy = -\int_{r=0}^{1}\int_{\theta=0}^{2\pi} r^2\sin^2\theta\cdot r\,dr\,d\theta = -\frac{\pi}{4}$$

Next, $\operatorname{div}\mathbf{F} = z^2+y^2+x^2$,

$$\iiint_A \operatorname{div}\mathbf{F}\,dV = \iiint_A (x^2+y^2+z^2)\,dV = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2}\int_{\rho=0}^{1}\rho^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{2\pi}{5}$$

Thus $$\iint_S \mathbf{F}\bullet\mathbf{n}\,dS = \iint_{S_2}\mathbf{F}\bullet\mathbf{n}\,dS - \iint_{S_1}\mathbf{F}\bullet\mathbf{n}\,dS = \iiint_A \operatorname{div}\mathbf{F}\,dV - \iint_{S_1}\mathbf{F}\bullet\mathbf{n}\,dS$$

$$= \frac{2\pi}{5} - \left(-\frac{\pi}{4}\right) = \frac{13\pi}{20}.$$