1. Use Lagrange multipliers to find the absolute maximum and minimum values of $z = x^3y$ on the circle $x^2 + y^2 = 4$. Find not only the values of z, but also the points (x, y) where they are attained.

Solution: *By the Extreme Value Theorem, absolute max and min values exist. The Lagrange multiplier equations are*

$$\begin{aligned} 3x^2y &= 2\lambda x && (1) \\ x^3 &= 2\lambda y && (2) \\ x^2 + y^2 &= 4. && (3) \end{aligned}$$

Assuming $xy \neq 0$, $\dfrac{3x^2y}{x} = 2\lambda = \dfrac{x^3}{y}$, *so* $x^2 = 3y^2$. *Substituting in (3) gives* $y^2 = 1$, *and the four points:* $(\pm\sqrt{3}, \pm 1)$, $(\pm\sqrt{3}, \mp 1)$ *with* $z = \pm 3\sqrt{3}$. *In addition the constraint (3) implies that* x *and* y *can't both be* 0, *so by equation (2),* $y \neq 0$. *But it is possible that* $x = 0 = \lambda$ *and* $y^2 = 4$. *However, these points give* $z = 0$. *Thus the maximum is* $z = 3^{3/2}$, *attained at* $(\pm\sqrt{3}, \pm 1)$, *and the minimum is* $z = -3^{3/2}$, *attained at* $(\pm\sqrt{3}, \mp 1)$.

2. Find all the critical points of $f(x, y) = 2x^3y - 6xy + y^2$, and classify them as local maxima, local minima or saddle points.

Solution: *The critical points are the solutions of*

$$\begin{aligned} f_x &= 6x^2y - 6y = 0 \\ f_y &= 2x^3 - 6x + 2y = 0 \end{aligned}.$$

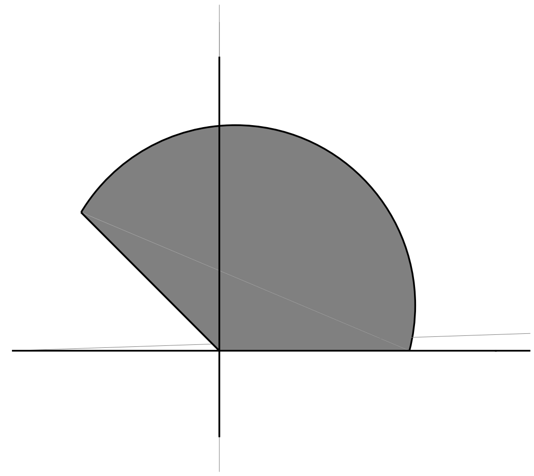
From $f_x = 0$ *we get* $y(x^2 - 1) = 0$, *so* $y = 0$ *or* $x = \pm 1$. *If* $y = 0$, *then* $f_y = 0$ *leads to* $x^3 - 3x = 0$, *giving three critical points:* $x = 0$ *or* $\pm\sqrt{3}$, *with* $y = 0$. *On the other hand if* $x = 1$, *then* $f_y = 0$ *leads to* $y = 2$, *and similarly* $x = -1$ *leads to* $y = -2$. *In all there are 5 critical points:* $(0, 0)$, $(\pm\sqrt{3}, 0)$, $(1, 2)$, *and* $(-1, -2)$.

Next, $A = f_{xx} = 12xy$, $B = f_{xy} = 6x^2 - 6$, *and* $C = f_{yy} = 2$. *Hence at* $(0, 0)$ *and* $(\pm\sqrt{3}, 0)$, $AC - B^2 = -B^2 < 0$, *so these are saddle points. On the other hand at* $(1, 2)$ *and* $(-1, -2)$, $A > 0$ *and* $AC - B^2 > 0$, *so these two points are local minima.*

3. Evaluate $\displaystyle\iint_R 5y\,dA$ where R is the region above the x-axis, above the line $y = -x$ and inside the disk $x^2 + y^2 \le 6$.

Solution:

In polar coordinates, R is described by $0 \le r \le \sqrt{6}$ and $0 \le \theta \le 3\pi/4$. The integral is



$$\int_{r=0}^{\sqrt{6}} \int_{\theta=0}^{3\pi/4} 5r\sin\theta \cdot r\,dr\,d\theta$$

$$= 5\int_0^{\sqrt{6}} r^2\,dr \cdot \int_0^{3\pi/4} \sin\theta\,d\theta$$

$$= \frac{5}{3} 6^{3/2} \left(\frac{\sqrt{2}}{2}\right).$$

4. Consider $\displaystyle\iiint_S x^2yz\,dV$ where S is the quadrant of the ball $x^2 + y^2 + z^2 \le A^2$ for which $y \ge 0$ and $z \ge 0$. (Here A is a positive constant.)

(a) Set up the integral in cylindrical coordinates.

Solution:

$$\iiint_S x^2yz\,dV = \int_{\theta=0}^{\pi} \int_{r=0}^{A} \int_{z=0}^{\sqrt{A^2-r^2}} (r\cos\theta)^2(r\sin\theta)(z)r\,dz\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi} \int_{r=0}^{A} \int_{z=0}^{\sqrt{A^2-r^2}} r^4 z\cos^2\theta\sin\theta\,dz\,dr\,d\theta$$

for (c) $$= \int_{\theta=0}^{\pi} \int_{r=0}^{A} \frac{1}{2} r^4 z^2 \cos^2\theta \sin\theta \Big|_{z=0}^{z=\sqrt{A^2-r^2}} dr\,d\theta$$

$$= \int_{\theta=0}^{\pi} \int_{r=0}^{A} \frac{1}{2} r^4 (A^2 - r^2)\cos^2\theta\sin\theta\,dr\,d\theta$$

$$= \frac{1}{2}\int_{r=0}^{A} (r^4A^2 - r^6)\,dr \cdot \int_0^{\pi} \cos^2\theta\sin\theta\,d\theta = \frac{A^7}{35}\cdot\frac{2}{3} = \frac{2A^7}{105}$$

(b) Set it up in spherical coordinates as well.

Solution:

$$\iiint_S x^2yz\,dV = \int_{\theta=0}^{\pi}\int_{\rho=0}^{A}\int_{\phi=0}^{\frac{\pi}{2}} (\rho\sin\phi\cos\theta)^2(\rho\sin\phi\sin\theta)(\rho\cos\phi)\rho^2\sin\phi\,d\phi\,d\rho\,d\theta$$

$$= \int_{\theta=0}^{\pi}\int_{\rho=0}^{A}\int_{\phi=0}^{\frac{\pi}{2}} \rho^6\cos^2\theta\sin\theta\sin^4\phi\cos\phi\,d\phi\,d\rho\,d\theta$$

$$\textit{for (c)} \quad = \int_0^{\pi}\cos^2\theta\sin\theta\,d\theta\cdot\int_0^{A}\rho^6\,d\rho\cdot\int_0^{\frac{\pi}{2}}\sin^4\phi\cos\phi\,d\phi = \frac{2}{3}\cdot\frac{A^7}{7}\cdot\frac{1}{5} = \frac{2A^7}{105}$$

(c) Evaluate one of these integrals.

Solution: *See above. The substitutions* $u=\cos\theta$ *and* $v=\sin\phi$ *were used.*

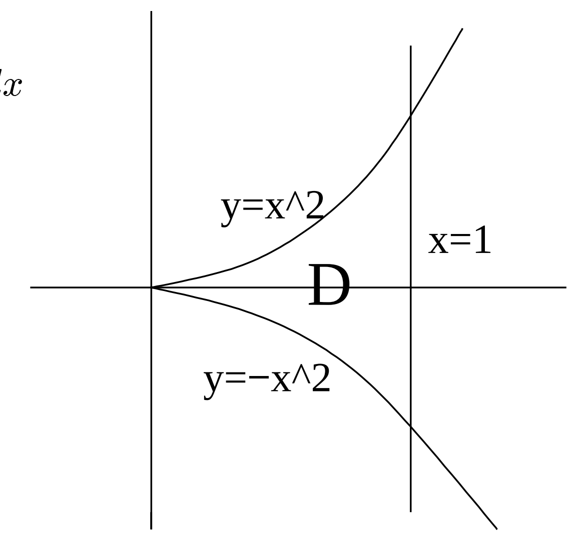
5. A plane lamina, sitting in the x,y-plane, lies between the parabolas $y = x^2$ and $y = -x^2$, and is bounded by the line $x = 1$. The density of the lamina at (x, y) is $\delta(x, y) = x^3$.

(a) Find the mass of the lamina.

Solution:

$$m = \iint_D \delta(x,y)\,dA = \int_{x=0}^{1}\int_{y=-x^2}^{x^2} x^3\,dy\,dx$$

$$= \int_{x=0}^{1} x^3y\Big|_{y=-x^2}^{y=x^2}\,dx = \int_0^1 2x^5\,dx = \frac{2}{6}.$$



(b) Find the center of mass of the lamina.

Solution: $\overline{y} = 0$ *since both* D *and* δ *are symmetric about the* x*-axis, and*

$$m\overline{x} = \iint_D x\delta(x,y)\,dA = \int_{x=0}^{1}\int_{y=-x^2}^{x^2} x\cdot x^3\,dy\,dx = \int_0^1 2x^6\,dx = \frac{2}{7}.$$

Therefore $\overline{x} = \dfrac{2/7}{2/6} = \dfrac{6}{7}$, *so* $(\overline{x},\overline{y}) = \left(\dfrac{6}{7}, 0\right)$.

6. Consider the surface $z^3 - 2x^2yz^2 + x^4y^2 = 4$.

(a) As (x, y, z) varies on the surface near the point $(1, -1, -1)$, z is a function of x and y. Give general formulas for $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ valid near $(1, -1, -1)$.

Solution: *Let* $F(x,y,z) = z^3 - 2x^2yz^2 + x^4y^2$. *Then*
$\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{-4xyz^2 + 4x^3y^2}{3z^2 - 4x^2yz}$ *and* $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{-2x^2z^2 + 2x^4y}{3z^2 - 4x^2yz}$.

(b) Find an equation for the tangent line to this surface at $(1,-1,-1)$.

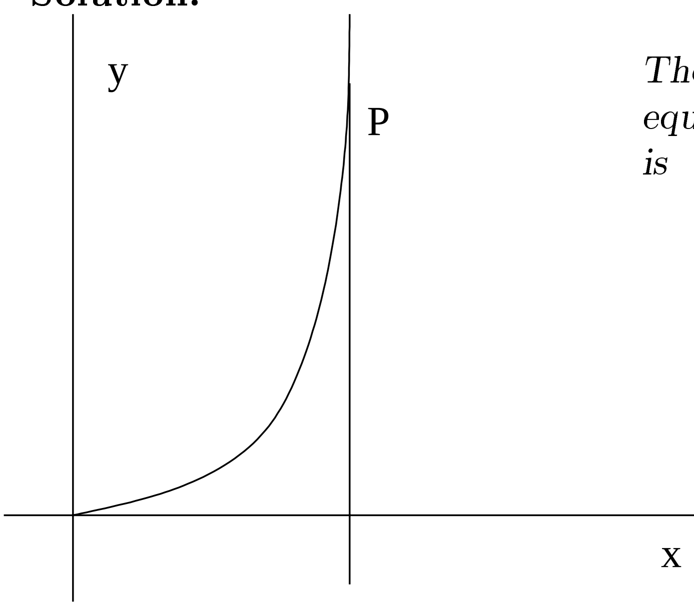
Solution: *The surface is a level surface of F so a normal vector is*
$\vec{\nabla}F\big|_{(1,-1,-1)} = \big\langle -4xyz^2 + 4x^3y^2, -2x^2z^2 + 2x^4y, 3z^2 - 4x^2yz\big\rangle\big|_{(1,-1,-1)} = \langle 8,-4,-1\rangle$.
The tangent plane has equation $8x - 4y - z = 13$.

7. Evaluate $\displaystyle\int_C 2x^2\,dx + xy\,dy$, where C is the (directed) line segment from $(2,3)$ to $(3,5)$. **Solution:** *The line between* $(2,3)$ *and* $(3,5)$ *has equation* $y = 2x-1$. *Since different points of C have different x-coordinates we can use $t = x$ as a parameter. Then C is parametrized as $x = t$, $y = 2t-1$. Thus $dx = dt$ and* $dy = \dfrac{dy}{dt}\,dt = 2\,dt$. *Therefore*

$$\int_C 2x^2\,dx + xy\,dy = \int_2^3 \big(2t^2 + t(2t-1)\cdot 2\big)\,dt = \int_2^3 6t^2 - 2t\,dt = 33.$$

8. Write $\displaystyle\int_{x=0}^{5}\int_{y=0}^{5x^2} e^{\cos x}\,dy\,dx$ as an equivalent iterated integral with the order of integration reversed. DO NOT EVALUATE.

Solution:



The point P is $(5,5^3)$. Also for $x \ge 0$, $y = 5x^2$ is equivalent to $x = \sqrt{y/5}$. The equivalent integral is

$$\int_{y=0}^{5^3}\int_{x=\sqrt{y/5}}^{5} e^{\cos x}\,dx\,dy.$$