Mathematics 251 Maple Lab 3

Maximum and Minimum Values

Fall 2006

This project Please turn in only the printout of your Maple worksheet. Use the **text** feature of Maple to add a header containing your name at the top of the worksheet, and **discussion** items where needed throughout the worksheet. Use the **title** option in all plots to introduce a label that will be kept with the plot when your report is printed.

The worksheet in the **seed file** is divided into **Sections** corresponding to the parts of this project description. It also contains almost all you need for problems 0 and 1; and imitating those instructions should allow you to complete the other parts. You may elaborate on this organization in preparing your report. Also, remove from the worksheet any extraneous material and any errors you have made.

In this lab, we use *Maple* to help visualize and compute the maximum and minimum values of a function of two variables. The functions appearing in this lab, like those appearing elsewhere in the course, are usually polynomials, with a few appearances of exponential or trigonometric functions. The names of these standard functions and the usual notation of algebra allow you to write the expressions for their values to which you apply the rules of calculus. The functions will be described here in standard mathematical notation. You will need to supply the translation into Maple idioms. For example, e^{2x^2} must be written as `exp(2*x^2)` or `exp(2*x*x)`.

Problem 0: Setup As in Lab 2, we begin by loading the `plots` library and fixing some options. Note that, in contrast to earlier work, we replace the `scaling=CONSTRAINED` option with `scaling=UNCONSTRAINED` since it is not necessary to compare distances along different axes (this is the default setting, so it could be omitted, but it is included to restore this setting if it had been changed). Then we introduce the expressions

$$\frac{y^3}{9} + 3x^2y + 9x^2 + y^2 + xy + 9 \qquad \text{(A)}$$

$$x(1 - 9xy)e^{-x^2-3y^2} \qquad \text{(B)}$$

that will be studied in this project (you should recognize (B) from problem 7 of Lab 0, where it was plotted over $-4 \le x \le 4$, $-4 \le y \le 4$). Although the discussion of this topic in Calculus uses **functions**, it is easier to work with **expressions** (or **names** like `A` or `B` that evaluate to those expressions) in Maple. Throughout this worksheet, x and y will be treated as independent variables. All other names will stand for expressions depending on those variables. That is, they are the **values** of certain functions of x and y. Thus, we use the name `A` for the expression in (A) and `B` for the expression in (B). We also introduce `R` for the region $\mathcal{R}$ that is the elliptical disk $9x^2 + (y+4)^2 \le 9$, and `bR` for an expression that is zero on boundary of $\mathcal{R}$, positive outside $\mathcal{R}$, and negative inside $\mathcal{R}$. These quantities will be used in part 3 (Note the use of the name `bx` for an intermediate expression to simplify the description.) This description of $\mathcal{R}$ is based on expanding and factoring to obtain $9x^2 \le -7 - 8y - y^2 = -(7+y)(1+y)$ and solving for x. The partial derivatives of A and B are also found here. These derivative are needed often in this project, so they need names, and `Ax` is a convenient abbreviation for $\partial A/\partial x$.

```
with(plots):with(VectorCalculus):
setoptions3d(axes=BOXED,scaling=UNCONSTRAINED,style=PATCH);
SetCoordinates( 'cartesian'[x,y] );
DetM:=M->LinearAlgebra:-Determinant(M);
A:=y^3/9+3*x^2*y+9*x^2+y^2+x*y+9;
B:=x*(1-9*x*y)*exp(-x^2-3*y^2);
bx:=sqrt(-(7+y)*(1+y))/3;
R:=y = -7 ..  -1, x = -bx .. bx;
bR:=9*x^2+(y+4)^2-9;
GA:=Gradient(A);
GB:=simplify(Gradient(B));
```

The supplementary worksheet includes instructions illustrating the need for the `simplify` in the definition of `GB` as well as a comparison of the use of `Gradient` (part of the `VectorCalculus` library) with direct computation of partial derivatives.

Problem 1: Local properties of critical points In this problem we wish to find and classify all the critical points of the expression (A).

(a) First obtain a rough idea of what this function looks like, by plotting it over the region $-1.5 \le x \le 1.5$, $-7 \le y \le 1$. You should be able to construct the graph without any hints, but be sure to use the `title` option to label it and give an explicit `orientation` option to assure that the plot will always be constructed as you want it to be. Examine the graph for possible critical points, but keep your observations to yourself. The results that you calculate later may turn out to be different from what you expect.

(b) Now use Maple's `solve` command to find all the critical points of the expression (A) using the instruction `solA:=[solve(convert(GA,set))];` . Note that, if you ask *Maple* to solve an **expression** or a set of expressions, it assumes that you want to solve the equations in which all expressions are equal to zero. To apply this to the entries of a `Vector`, a `set` was formed from the entries using the `convert` instruction. It is also a good idea to name the solutions when they are computed, so we assign the result the name `solA`. The solutions obtained from the `solve` command are given to you as an **expression sequence**, i.e., several quantities separated by commas (as in the definition of $\mathcal{R}$ above). In this case, the elements in the sequence are sets of assignments of the variables. To get a more useful result, the sequence is enclosed in brackets to turn it into a `list`. Then you can retrieve individual solutions using the usual *Maple* indexing convention, e.g., `solA[3]` refers to the third one. To check this, you can evaluate the vector `GA` at `solA[3]` by typing `eval(GA,solA[3]);`. The supplementary worksheet contains this example. Although this won't appear in your report, make sure you understand why the result is **not** surprising. (The `subs()` instruction may be used instead of `eval()`. The order in which the arguments appear differs, but the results are similar. Since `eval()` is more useful for our applications, lab descriptions and seed files encourage its use.)

(c) The `Hessian` command from the `VectorCalculus` package produces the matrix of second derivatives of a function (see the **help and glossary pages** for details) The textbook (see p. 954) classifies critical points using a second derivative test involving a quantity

$$D = \frac{\partial^2 A}{\partial x^2} \cdot \frac{\partial^2 A}{\partial y^2} - \frac{\partial^2 A}{\partial x\, \partial y} \cdot \frac{\partial^2 A}{\partial y\, \partial x}$$

that is the **determinant** of the **Hessian matrix**. A function to compute determinants has been imported at the start of the worksheet to allow the determinant to be found easily. **Compute the expression** D (using the name `DHA` appearing in the seed file) and **evaluate it** in your main worksheet, together with **anything else that you need to classify the critical point**, at **each** critical point. When you have these values, add a line or two of **text** giving the type of each of the four critical points. (You should be able to this without any hints, so the seed file contains only an outline to give a place to put results for each critical point.)

Problem 2: Global extrema

Now consider the expression B. If you don't have the graph from Lab 0 handy, recreate it in the supplementary worksheet. Don't include it in your main worksheet

(a) First, use the `solve` command to find critical points as in problem 1 and assign the name `solB` to the result. The solution (or part of the solution) will be given in terms of an algebraic equation. The degree of this equation usually gives the number of roots. Add a **discussion** identifying this degree and comparing it to the number of **visible** critical points. If there are additional algebraic roots, include an explanation of the difference.

There is space in the supplementary worksheet for experiments to support you discussion, but you will need to supply the instructions that your version of Maple will accept. This is due to version-dependent changes in the `solve` function. In Maple 9, the answer was given as a single `RootOf` expression, but Maple 10 gives a second solution where the exponential factor is zero (because Maple allows complex number solutions to all equations). Once you have extracted a single `RootOf` expression, the `allvalues` function will extract a set of symbolic expressions for the roots. If you feed this result into `evalf` you can get numerical approximations. Because we cannot predict what will be necessary to find roots by this method, or how the real roots will be recognized, a different method of finding the critical points in the remainder of this problem. However, the `allvalues` function will appear again in problem **3c**.

(b) An alternative is to use the `fsolve` command to find the critical points that lie in the region $-2 \le x \le 2$, $-2 \le y \le 2$. There are exactly four of them, as indicated by the plot from Lab 0. You should also find the values of the function the expression (B) at all of these critical points; be sure that your worksheet makes it clear which values are obtained at which points. The `fsolve` command uses an iterative method (such as Newton's method) to find the roots and sometimes the method does not converge to a root. You should consult the help file for the `fsolve` command to find how to restrict the search for a root to a smaller region, and use your plot to identify suitable regions. The seed file contains the line `p1:=fsolve(convert(GB,set), {x,y}, x =-1..0, y=0..1);` that finds one of the critical points and assigns it the name `p1` (having previously defined `GB` as the simplification of the gradient of `B`. An efficient way to evaluate B at this point using `eval` is also included in the seed file. Use this as a model for finding the value of B at all critical points.

(c) This function is close to zero if $x^2 + y^2$ is large, and the plot reveals that it takes both positive and negative values. The maximum and minimum must be attained at critical points, and you should now know, and have names for, all of them to reasonable accuracy. Use this to determine the absolute minimum and absolute maximum values of the expression (B). Summarize in **text**.

Problem 3: Constrained extrema

In this problem, we find the absolute minimum and absolute maximum of the expression (A) of Problem 1 on the ellipse $\mathcal{R}$ whose description was given at the beginning of this Lab (and included in the seed file). We know from the general theory that the absolute minimum and maximum of the expression A occur either (1) at critical points of the expression A which lie in the interior of the region $\mathcal{R}$ or (2) on the boundary of the region $\mathcal{R}$. The boundary is a smooth arc on which the method of **Lagrange Multipliers** applies.

(a) Obtain a plot of the expression A with the domain restricted to the region $\mathcal{R}$. The supplementary worksheet contains an instruction that will produce the plot. You must supply a title and suitable `orientation` before copying the instruction to your main worksheet. The choice of `orientation` should illustrate the determination of the extreme values and the points at which they are attained.

(b) Determine which of the critical points found in Problem 1 lie in the region $\mathcal{R}$ and evaluate the expression A at these points. The easiest way to do this is to evaluate $bR = 9x^2 + (y+4)^2 - 9$ at the values named by each `solA[i]`. Those points giving **negative** values are **inside** the ellipse. An example of this is included in the seed file. State your conclusions in a **text** section, but leave the calculations supporting your conclusion in the worksheet. (An alternative would be follow the definition of R by substituting the y value at each point into `bx` and determine whether the corresponding x value lies between this quantity and its negative. The supplementary worksheet should be used for such work. Do not copy such explorations into the main worksheet. Only the testing of $9x^2 + (y+4)^2 - 9$ at critical points should be shown.)

(c) For functions of two variables, the criterion for a extreme value of $f(x, y)$ subject to the constraint $g(x, y) = 0$ specializes to $f_x g_y = f_y g_x$ when the **Lagrange multiplier** is eliminated. Solving this simultaneously with $g(x, y) = 0$ determines the points that must be considered. The supplementary worksheet contains instructions to create a matrix whose rows are the two gradients (converted to `Vectors` to allow the matrix to be built) and find its determinant to test whether the vectors are parallel. The result of these instructions is an **algebraic** solution that is not immediately useful for our purpose. However, the method used in problem **2a** will yield a list of all **complex number** solutions of the equation.

Create this list and select the **real number** solutions in it. Then, evaluate A at these points. Insert a **text** summary of your results in the main worksheet, including approximate values of the extreme values on the boundary and the location of those extrema.

(d) Combine the results of (b) and (c). You now have a list of all possible locations of maxima and minima on $\mathcal{R}$ (both interior points and boundary points) and the value of A at these points. By identifying the smallest and largest values of A in this list, you will have found the extreme values on $\mathcal{R}$ and point where those values are attained. You will **not** need the classification of critical points from problem **1**, although the results should be **consistent** (i.e. a **global** maximum or minimum at an interior point must be a local extremum of the same type). The graph in part 3a should also be consistent with your results.

Use **text** to state your conclusions.

End of Lab 3