Please note that this is *not* a practice exam; in particular, there are more problems here than will be on the exam. Moreover, this set of problems does not provide exhaustive coverage of all material in the course; you should study also your earlier exams and the corresponding review problems. Finally, although these problems are generally similar to exam problems, it is possible that the exam will contain some problems quite different from any here. The answers are not guaranteed.

1. Find the curl and divergence of the following vector fields $\mathbf{F}$. If the field is conservative, find a function f so that $\mathbf{F} = \nabla f$.
(a) $\mathbf{F}(x,y,z) = \sin x\mathbf{i} + \cos x\mathbf{j} + z^2\mathbf{k}$ $\quad$ $\text{curl}\,\mathbf{F}{=}-\sin x\mathbf{k},\quad \text{div}\,\mathbf{F}{=}\cos x{+}2z$
(b) $\mathbf{F}(x,y,z) = z\mathbf{i} + 2yz\mathbf{j} + (x + y^2 - z^2)\mathbf{k}$ $\quad$ $\text{curl}\,\mathbf{F}{=}0,\quad \text{div}\,\mathbf{F}{=}0,\quad f{=}xz{+}y^2z{-}(1/3)z^3$

2. A vector field $\mathbf{F}$ is called *irrotational* if $\text{curl}\,\mathbf{F} = 0$, and it is called *incompressible* if $\text{div}\,\mathbf{F} = 0$.
(a) Show that any vector field of the form $\mathbf{F}(x,y,z) = f(x)\mathbf{i} + g(y)\mathbf{j} + h(z)\mathbf{k}$ is irrotational.
(b) Show that any vector field of the form $\mathbf{F}(x,y,z) = f(y,z)\mathbf{i} + g(x,z)\mathbf{j} + h(x,y)\mathbf{k}$ is incompressible.
(c) Suppose ϕ is any scalar function and $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is any vector field. Show that

$$\text{curl}(\phi\,\mathbf{F}) = \phi\,\text{curl}\,\mathbf{F} + (\nabla\phi) \times \mathbf{F}, \qquad \text{div}(\phi\,\mathbf{F}) = \phi\,\text{div}\,\mathbf{F} + (\nabla\phi)\cdot\mathbf{F}$$

(d) Suppose ϕ and ψ are any scalar functions. Use part (c) to show that

$$\text{curl}(\phi\,\nabla\psi) = (\nabla\phi) \times (\nabla\psi).$$

In particular, show that $\phi\nabla\phi$ is irrotational.
(e) Suppose ϕ and ψ are any scalar functions. Use part (c) to show that

$$\text{div}(\phi\,\nabla\psi) = (\text{div}\,\phi)\cdot(\text{div}\,\psi).$$

In particular, show that if $\phi\nabla\phi$ is incompressible, then ϕ is constant.

3. Let E be the solid region $3 \le z \le 4 - (x^2 + y^2)$, S its boundary surface, and $\mathbf{F}$ the vector field $\mathbf{F} = (y + x)\mathbf{i} + (y - x)\mathbf{j}$. The divergence theorem asserts that

$$\iiint_E \text{div}\,\mathbf{F}\,dV = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS,$$

where $\mathbf{n}$ is the outward unit normal vector on S and dS is the element of surface area. Verify this for $\mathbf{F}$ and E by calculating both sides and showing that they are equal. $\pi{=}\pi$

4. Let $\mathbf{a}(x,y,z)$ be the vector field $\mathbf{a} = (x + y)\mathbf{i} + (y - x)\mathbf{j} + z\mathbf{k}$.
(a) Evaluate $\int_\Gamma \mathbf{a}\cdot d\mathbf{r}$ for Γ the straight line segment joining (0,1,2) to (-1,2,0). -1
(b) Evaluate $\oint_C \mathbf{a}\cdot d\mathbf{r}$ for C the circle $x^2 + y^2 = 4$, oriented *clockwise*, in the plane $z = 0$. 8π
(c) Let S be the hemisphere $x^2 + y^2 + z^2 = 4$, $z \le 0$. Stokes' Theorem asserts that

$$\int_C \mathbf{a}\cdot d\mathbf{r} = \iint_S \text{curl}\,\mathbf{a}\cdot\mathbf{n}\,dS$$

where $\mathbf{n}$ is the normal vector to S pointing in the positive z direction. Calculate the surface integral in this case and verify that it agrees with the answer in (b). $8\pi{=}8\pi$

5. Let S_1 be the disk $x^2+y^2 \le 1$ in the plane $z=0$, let S_2 be the hemisphere $x^2+y^2+z^2=1$, $z \ge 0$, and let $\mathbf{F}$ be a twice differentiable vector field defined in all of space.
(a) Show that Stokes' Theorem implies that

$$\iint_{S_1} \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_2} \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS, \qquad (**)$$

where the unit normal vectors $\mathbf{n}$ are chosen to have positive z component ($\mathbf{n}\cdot\mathbf{k}>0$).
(b) Show that $\operatorname{div}\operatorname{curl}\mathbf{F}=0$.
(c) Derive the equality $(**)$ from the result of part (b) and the Divergence Theorem.

6. Let C be the triangle in the xy plane with vertices at (0,0), (1,0), and (0,1), oriented counterclockwise.
(a) Evaluate $\oint_C 2y^2\,dx + 2x\,dy$ directly as a line integral. 1/3
(b) Evaluate the line integral in (a) by using Green's Theorem and then evaluating a double integral.

7. Let R be the region in the first quadrant bounded by the lines $y=x$, $y=3x$ and the hyperbolas $xy=1$, $xy=3$. Use the transformation $x=u/v$, $y=v$ to evaluate the integral $\int\int_R xy\,dA$. $2\ln 3$

8. Compute $\iiint_D (x^2+y^2)\,dV$, where D is the region inside the hemisphere $x^2+y^2+z^2=9$, $z\ge 0$, and between the cones $z^2=x^2+y^2$ and $z^2=3(x^2+y^2)$. Be sure to use the easiest coordinate system. $81/20(9\sqrt{3}-10\sqrt{2})\pi$

9. Find the surface area of the portion of the surface $z-x^2-y^2=9$ which lies between the planes $z=10$ and $z=13$. $\pi(17^{3/2}-5^{3/2})/6$

10. For the integral $\displaystyle\int_0^4\int_{x^2-4x}^{x}(x+2y)\,dy\,dx$: sketch the region of integration, evaluate the integral, and write down the integral with the reversed order of integration.
$448/15;\ \int_{-4}^{0}\int_{2-\sqrt{y+4}}^{2+\sqrt{y+4}}(x+2y)\,dx\,dy + \int_0^4\int_y^4 (x+2y)\,dx\,dy$

11. Let R be the region for which $(x-4)^2+y^2\le 16$ and $x\ge y$. Evaluate $\iint_R xy\,dA$ using polar coordinates. $-64/3$

12. Let $f(x,y,z)=xy+yz+zx$, and let P be the point $P(1,2,3)$.
(a) Find the direction in which the function is increasing the most rapidly at the point P, and the corresponding rate of increase. $\langle 5,4,3\rangle/\sqrt{50};\ \sqrt{50}$
(b) Find one direction in which the directional derivative of the function at P is 0. $(-4\mathbf{i}+5\mathbf{j})/\sqrt{41}$
(c) Find the directional derivative of f in the direction of $\mathbf{b}=2\mathbf{i}+4\mathbf{j}-10\mathbf{k}$ at P. $-2/\sqrt{30}$
(d) Find the equations of the tangent plane and normal line to the surface $f(x,y,z)=11$ at P.
(e) Use differentials to estimate $f(1.1,1.8,3.2)$. (d) $5x+4y+3z=22$; $x=1+5t$, $y=2+4t$, $z=3+3t$; (e) 11.3

13. Find the absolute maximum and minimum values of $f(x,y)=x^2-6x+y^2-8y+5$ in the region $x^2+y^2\le 36$. (Hint: you must look for critical points and also check for extreme values on the boundary.) Min: -20 at $(3,4)$, Max: 101 at $(-18/5,-24/5)$

14. Suppose that $w=x^2y-y^3$ with $x=f(r,s)$, $y=g(r,s)$. Find $\left.\dfrac{\partial w}{\partial r}\right|_{\substack{r=2\\ s=1}}$ and $\left.\dfrac{\partial^2 w}{\partial r\partial s}\right|_{\substack{r=2\\ s=1}}$, given

$$f(2,1)=1,\quad g(2,1)=-2\quad f_r(2,1)=2,\quad f_s(2,1)=-1\quad g_r(2,1)=3,\quad g_s(2,1)=-2$$
$$f_{rr}(2,1)=4,\quad f_{rs}(2,1)=2\quad f_{ss}(2,1)=2\quad g_{rr}(2,1)=-3,\quad g_{rs}(2,1)=0,\quad g_{ss}(2,1)=4$$

$-41;\ -86$

15. A particle moving in space has acceleration at time t given by $\mathbf{a}(t)=2\mathbf{i}-\mathbf{j}+3t\mathbf{k}$. The particle passes through the origin at time $t=0$ with velocity $-\mathbf{i}+2\mathbf{j}$. Find its position at time t and its velocity as it passes through the point $(2,2,4)$. $\mathbf{r}=\langle t(t-1), t(2-t/2), t^3/2\rangle;\mathbf{v}=\langle 3,0,6\rangle$